

Energy-Efficient Congestion Control

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ABSTRACT

Various link bandwidth adjustment mechanisms are being developed to save network energy. However, their interaction with congestion control can significantly reduce throughput, and is not well understood. We firstly put forward an easily implementable link dynamic bandwidth adjustment (DBA) mechanism, and then study its iteration with congestion control. In DBA, each link updates its bandwidth according to an integral control law to match its average buffer size with a target buffer size. We prove that DBA reduces link bandwidth without sacrificing throughput—DBA only turns off excess bandwidth—in the presence of congestion control. Preliminary ns2 simulations confirm this result.

Categories and Subject Descriptors

C.2.1 [Computer-Communication Networks]: Network Architecture and Design—*Distributed networks*

Keywords

bandwidth adjustment, congestion control, stability.

1. INTRODUCTION

Energy efficiency is becoming one of the main design criterias in the information and communication industry, from hardware design (devices to systems) to software configuration (e.g., virtualization), and to service deployment. While significant advances in semiconductor technology have led to decrease in power consumption per byte transmission, the power efficiency is starting to plateau and the demand for greater bandwidth is leading to an overall increase in energy consumption [10]. The level of energy consumption in information and communication industry is growing exponentially at an annual rate of 15%. This trend could lead to significant cost increase to network operators [10, 13]. Therefore, there is a need to reduce energy consumption through other power management technologies. In fact, there have been ongoing efforts to develop mechanisms to reduce network energy consumption by opportunistically turning off or slowing down network devices during periods of low traffic loads. Sleep-mode operations have been proposed at various network layers, and dynamic resource adjustment is being facilitated by voltage and frequency scaling techniques [12].

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SIGMETRICS'12, June 11–15, 2012, London, England, UK.

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Today's core networks have large link bandwidths with low utilization levels (30-40%) [11], providing opportunities for turning off excess bandwidth and saving energy, especially when traffic loads follow diurnal pattern, and link utilization level is low during off-peak hours. Turning off excess bandwidth may be achieved by turning off one or more of the parallel links between routers, or slowing down a particular link. With current technology, power consumption in backbone routers and their line cards is essentially load-independent and therefore selective application of sleep mode is more feasible during periods of low utilization—[11, 13]. In core networks, pairs of routers are typically connected by multiple physical links, called a link bundle, and the bundle is used as a logical link for routing purposes. Link bundling, standardized in IEEE 802.1AX [11], facilitates incremental bandwidth upgrade and prolongs the lifetime of a router. Furthermore, individual link components of a bundle may be selectively turned off to provide graceful bandwidth adjustment and energy savings during low traffic loads without affecting the topology. In the future, application of frequency and voltage scaling could facilitate fine-grain adjustment of the bandwidth of each individual link component.

TCP is the dominant transport layer protocol on the Internet, and all energy saving mechanisms in the network can interact with TCP in intricate ways. Link bandwidth adjustment can lead to serious TCP throughput reduction and instability if it is not done in concert with the dynamics of TCP and active queue management, which is often represented by random early discard (RED). The objective of the paper is to study energy-saving bandwidth adjustment mechanisms in the presence of TCP/RED.

The contribution of this paper are mainly three folds. First, we propose a dynamic bandwidth adjustment (DBA) mechanism that updates the bandwidth with local buffer information, thus easy to implement. Each link uses an integral control law to match its average buffer size with some moving target buffer size. The target buffer size is a function of the bandwidth, and corresponds to a chosen queueing delay. Second, we prove the following results regarding the joint dynamic of TCP/RED/DBA.

- TCP/RED/DBA is globally stable if network delay is negligible. At steady state, excess bandwidth is turned off, and throughput is preserved.
- The steady-state TCP source rates and link bandwidths solve an optimization problem, which we call the network surplus maximization problem.
- TCP/RED/DBA is locally stable if network delay is bounded and link dynamics are sufficiently slow.

Third, we evaluate the performance of DBA through ns2 simulations, and confirm that DBA turns off excess bandwidth without reducing throughput.

The rest of the paper is structured as follows. In section

2, we introduce some preliminaries including the network model and the utility maximization framework. In section 3, we propose a dynamic bandwidth adjustment (DBA) mechanism, and give its implementations. We analyze the steady state and transient stability of TCP/RED/DBA in section 4, and provide ns2 simulations in section 5.

2. PRELIMINARIES

Consider a network that consists of a set $\mathcal{L} := \{1, \dots, L\}$ of unidirectional links and a set $\mathcal{N} := \{1, \dots, N\}$ of sources. For each link $l \in \mathcal{L}$, let \mathcal{N}_l denote the set of sources that traverse it. For each source $i \in \mathcal{N}$, let \mathcal{L}_i denote the set of links on its path to the destination. Then $i \in \mathcal{N}_l$ is equivalent to $l \in \mathcal{L}_i$ for any $i \in \mathcal{N}$ and any $l \in \mathcal{L}$. We make the following assumptions on the links and sources.

Assumption 1. *Each link $l \in \mathcal{L}$ can set its bandwidth c_l to any value in the range $\mathcal{C}_l := [c_l^{min}, c_l^{max}]$.*

Link bandwidth may be adjusted by assuming that the link is in fact a link bundle, as described in the introduction, and an individual link component may be put to sleep or turned off to control the total bandwidth of the bundle. Alternatively, the bandwidth of a link may be adjusted with voltage and frequency scaling technology. We assume continuous bandwidth levels in the theoretical framework, and discuss discrete bandwidth levels in the implementation in section 3.2, and in the simulations in section 5.

Assumption 2. *Each source $i \in \mathcal{N}$ can set its rate x_i to any value in the range $\mathcal{X}_i := [0, x_i^{max}]$.*

Each source i is associated with a transmission window w_i , and transmits roughly w_i unacknowledged bits at any given time. Let τ_i denote the round-trip time (RTT) for source i , then source rate x_i is roughly given by $x_i = w_i/\tau_i$. Though TCP only controls x_i through adjusting w_i , we assume that TCP can control x_i explicitly as in [1, 2].

Define routing entries

$$R_{li} := \begin{cases} 1 & \text{if } i \in \mathcal{N}_l \\ 0 & \text{otherwise} \end{cases}, \quad l \in \mathcal{L}, \quad i \in \mathcal{N},$$

then the traffic y_l on link l is

$$y_l = \sum_{i \in \mathcal{N}_l} x_i = \sum_{i \in \mathcal{N}} R_{li} x_i, \quad l \in \mathcal{L}.$$

We use lower case letter without subscript to denote a column vector of the corresponding quantity. For example, x denotes $(x_1, \dots, x_N)^T$ and c denotes $(c_1, \dots, c_L)^T$.

Table 1: List of Symbols

i	a source, $i \in \mathcal{N} = \{1, \dots, N\}$
l	a link, $l \in \mathcal{L} = \{1, \dots, L\}$
c_l	capacity of link l
x_i	rate of source i
R_{li}	routing entry, $R_{li} = 1$ if source i passes link l , and 0 otherwise
y_l	traffic on link l
b_l	average buffer size on link l
p_l	packet drop probability on link l
q_i	end-to-end packet loss probability for source i
U_i	utility function of source i
$\bar{b}_l(c_l)$	target buffer size of link l
d_l	target queueing delay of link l
$\bar{p}_l(c_l)$	packet drop probability corresponding to \bar{b}_l
V_l	cost function of link l

2.1 Traditional Congestion Control

There are two types of congestion control: loss-based congestion control and delay-based congestion control, and we focus on loss-based congestion control in this paper. In loss-based congestion control, links drop packets to reflect their buffer occupancy, and sources adjust their transmission rates to respond to their end-to-end packet loss.

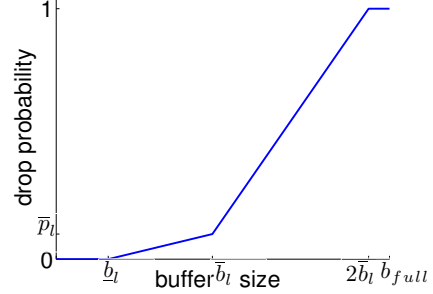


Figure 1: Packet drop probability for RED.

We consider active queue management link control, in particular the random early discard (RED) mechanism. In RED, a link l keeps track of its average buffer size b_l ,¹ and drops packets with probability (see Figure 1)

$$p_l = \begin{cases} 0 & \text{if } b_l \leq \underline{b}_l \\ \gamma_l(b_l - \underline{b}_l) & \text{if } \underline{b}_l < b_l \leq \bar{b}_l \\ \eta_l b_l - (1 - 2\bar{p}_l) & \text{if } \bar{b}_l < b_l \leq 2\bar{b}_l \\ 1 & \text{if } b_l > 2\bar{b}_l \end{cases},$$

where \underline{b}_l , \bar{b}_l , \bar{p}_l are design parameters and $\gamma_l = \frac{\bar{p}_l}{\bar{b}_l - \underline{b}_l}$, $\eta_l = \frac{1 - \bar{p}_l}{\bar{b}_l}$ are the corresponding slopes. As its buffer builds up, a link drops more and more packets, guiding the sources to reduce their transmission rates. In practice, TCP/RED often gets unstable when $b_l > \bar{b}_l$, and the parameters \bar{p}_l , \bar{b}_l , b_l are designed empirically or adaptively (ARED [14]) to avoid such instability. Hence, the model

$$p_l = \begin{cases} 0 & \text{if } b_l \leq \underline{b}_l \\ \gamma_l(b_l - \underline{b}_l) & \text{if } \underline{b}_l < b_l \leq \bar{b}_l \end{cases} \quad (1)$$

is sufficient for our study. In prior works (for example, [2]), the following simpler model

$$p_l = \gamma_l b_l \quad (2)$$

is often used. Model (2) neglects the region $[0, \underline{b}_l]$ on which $p_l = 0$, for the convenience of analysis. We will generally use model (2), but use the more precise model (1) to derive an important result: we can reduce the bandwidth without sacrificing throughput. Basically, after reducing the bandwidth, average buffer size b_l at non-bottleneck links increases. But if b_l is still below \underline{b}_l , the corresponding p_l remains zero. Consequently, sources will not observe additional packet loss, and throughput will be preserved.

We consider TCP source control. In TCP Reno, a widely used loss-based TCP, a source generally increases its window size by 1 packet per round-trip time and halves its window size after a packet loss. For each source i , the end-to-end

¹Average buffer size is the average of instantaneous buffer size over a moving time window of typically 500ms.

packet loss probability q_i it observes is given by

$$q_i = 1 - \prod_{l \in \mathcal{L}_i} (1 - p_l) \approx \sum_{l \in \mathcal{L}_i} p_l = \sum_{l \in \mathcal{L}} R_{li} p_l.$$

The approximation is due to the fact that $\sum_{l \in \mathcal{L}_i} p_l \ll 1$, i.e., in practice, the end-to-end packet loss probability is very small. At steady state, source i transmits at rate [7]

$$x_i = \begin{cases} x_i^{max} & \text{if } q_i < q_i^{min} \\ f_i(q_i) & \text{if } q_i \geq q_i^{min} \end{cases}, \quad (3)$$

where $f_i : [q_i^{min}, \infty) \rightarrow (0, x_i^{max}]$ is strictly decreasing, and determined by the TCP protocol source i uses. For example,

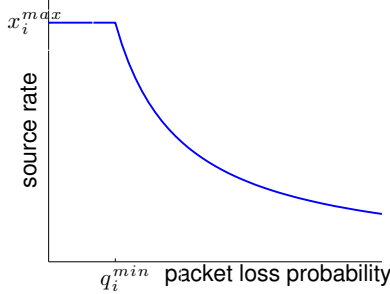


Figure 2: Source rate.

$f_i(q_i) = \frac{1}{\tau_i} \sqrt{\frac{3}{2q_i}}$ for TCP Reno, inversely proportional to the round-trip time τ_i , and decreases as loss q_i increases. Noting that (3) captures the stationary relationship between q_i and x_i for all TCP protocols, we use it for the sake of generality. Associate a utility function [7]

$$U_i(x_i) := \int^{x_i} f_i^{-1}(x) dx$$

with each source $i \in \mathcal{N}$, then (3) is equivalent to

$$x_i = \begin{cases} x_i^{max} & \text{if } q_i < U_i'(x_i^{max}) \\ U_i'^{-1}(q_i) & \text{if } q_i \geq U_i'(x_i^{max}) \end{cases}. \quad (4)$$

Since f_i^{-1} is strictly decreasing and positive for $x_i \leq x_i^{max}$, U_i is strictly concave and increasing. For instance, $U_i(x_i) = -\frac{3}{2\tau_i^2} \frac{1}{x_i}$ for TCP Reno [7].

2.2 Network Utility Maximization

For each link l , its average buffer size b_l updates as²

$$\dot{b}_l = \begin{cases} 0 & \text{if } b_l = 0 \text{ \& } y_l < c_l \\ y_l - c_l & \text{otherwise} \end{cases}. \quad (5)$$

Using model (2) for p_l , (5) is equivalent to

$$\dot{p}_l = \begin{cases} 0 & \text{if } p_l = 0 \text{ \& } y_l < c_l \\ \gamma_l(y_l - c_l) & \text{otherwise} \end{cases}. \quad (6)$$

Discrete version of (6) and (4) is proven to solve the following *network utility maximization* (NUM) problem [2]

$$\text{NUM} \begin{cases} \max_{x_i \in \mathcal{X}_i} & \sum_{i \in \mathcal{N}} U_i(x_i) \\ \text{s.t.} & y_l \leq c_l, l \in \mathcal{L}. \end{cases}$$

²We do not distinguish between instantaneous buffer size and average buffer size in this paper.

Let x^{NUM} denote the solution to NUM with $c_l = c_l^{max}$ for all l , then x^{NUM} is the steady-state source rate for TCP/RED. Let y^{NUM} denote the traffic corresponding to x^{NUM} .

3. BANDWIDTH ADJUSTMENT

In this section, we propose a dynamic bandwidth adjustment mechanism, and give its detailed implementations.

3.1 Algorithm

The objective of energy-efficient congestion control is to achieve the same throughput as traditional congestion control, while consuming less energy by turning off excess bandwidth. Conceptually, we can achieve this by finding x^{NUM} and reducing the bandwidth c to the traffic y^{NUM} . However, measuring the traffic on a link is expensive. In practice, a link only knows its own buffer size. To conclude, energy-efficient congestion control has the following objectives:

- achieves the source rate x^{NUM} ;
- reduces link bandwidth c to traffic rate y^{NUM} ;
- uses local buffer information to update c .

We maintain the existing TCP/RED architecture, and introduce bandwidth updates. Noting that the only available information is its buffer size, a link l compares its average buffer size b_l with a moving target $\tilde{b}_l(c_l)$, and updates its bandwidth proportional to their difference according to the following *dynamic bandwidth adjustment* (DBA) mechanism:

$$\dot{c}_l = \begin{cases} 0 & \text{if } c_l = c_l^{min} \text{ \& } b_l < \tilde{b}_l(c_l) \\ & \text{or } c_l = c_l^{max} \text{ \& } b_l > \tilde{b}_l(c_l) \\ \alpha_l (b_l - \tilde{b}_l(c_l)) & \text{otherwise} \end{cases}. \quad (7)$$

Design parameter $\alpha_l > 0$ characterizes bandwidth dynamic speed, and its value will be discussed in section 3.2. We can interpret (7) as a truncated integral control law with a moving target. If b_l exceeds \tilde{b}_l , bandwidth is increased to clear the buffer, and vice versa. At equilibrium, we expect $b_l = \tilde{b}_l(c_l)$. If the target \tilde{b}_l is a constant, then queueing delay generally increases as we decrease bandwidth, and quality of service can be degraded. In order to limit the queueing delay, \tilde{b}_l should decrease as c_l decreases. If throughput y_l is time invariant, the corresponding phase plot is shown in Figure 3: bandwidth c_l converges to the throughput y_l , and buffer size b_l equals its target $\tilde{b}_l(c_l)$ at convergence. However, y_l will react to link buffer dynamics due to TCP, calling for more detailed study, which is given in section 4.

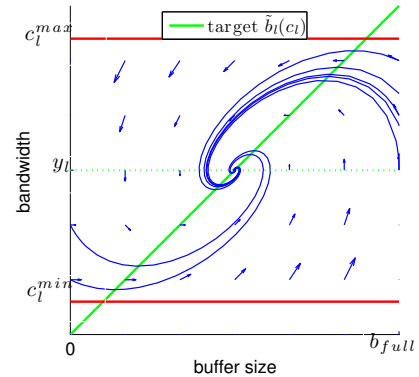


Figure 3: Phase plot of (b_l, c_l) for constant traffic y_l .

3.2 Implementation

In this section, we give the implementations of DBA. For ease of presentation, we consider a single link l . Recall that c_l , b_l , and $\tilde{b}_l(c_l)$ denote the bandwidth, average buffer size, and target buffer size of link l respectively. For notational brevity, we will drop the subscript l in this section.

We set $\tilde{b}(c) = dc$, where d is called *target queueing delay* and satisfies $d < b/c^{max}$, so that the moving target \tilde{b} is always below \underline{b} . Then, at steady state, packet drop probability $p_l = 0$ if link l is not at its full bandwidth.³ Consequently, either a link is at its full bandwidth, or it will not drop any packet. Hence, end-to-end loss probability q_i is not increased after we reduce the bandwidth. As a result, source rates won't be reduced. A rigorous statement is provided in Theorem 1. Noting that queueing delay on non-bottleneck links increases by d , we should set d small for quality of service consideration. A typical value is $d = 1ms$.

Algorithm 1 Continuous DBA

At epoch $k \in \mathbb{Z}$, input $c[k-1]$, $b[k-1]$.

1. Integral law.

```

if  $\left| \frac{b[k-1]}{dc[k-1]} - 1 \right| \leq \epsilon$  then
   $c \leftarrow c[k-1] + \delta_1 \frac{b[k-1] - dc[k-1]}{T}$ ;
else
   $c \leftarrow c[k-1] + \delta_2 \frac{b[k-1] - dc[k-1]}{T}$ ;
end if

```

2. Lower and upper bounds.

```

if  $c > c^{max}$  then
   $c \leftarrow c^{max}$ ;
else
  if  $c < c^{min}$  then
     $c \leftarrow c^{min}$ ;
  end if
end if

```

3. Switching.

```

 $c[k] \leftarrow c$ ;

```

Return $c[k]$ as the link bandwidth during $[kT, (k+1)T)$.

Bandwidth c should be updated periodically. Though current technology supports switching each physical interface (for instance, an optical cable) between idle and sleep states within 1ms, the updating period T should be much larger than 1ms to reduce switching overhead. On the other hand, the period T cannot be too large, in order to remain responsive to traffic increases. A typical value is $T = 100ms$. Assume bandwidth updates at epochs $t = kT$, $k \in \mathbb{Z}$, and define $c[k] := c(kT)$ as the bandwidth at epoch k (then $c[k]$ is also the bandwidth during $[kT, (k+1)T)$). Similarly, define $b[k] := b(kT)$, $\tilde{b}[k] := \tilde{b}(kT) = dc(kT)$ as the average buffer size and target buffer size at epoch k . If we discretize (7) and neglect the lower and upper bounds on c , we get

$$c[k+1] = c[k] + T\alpha(b[k] - \tilde{b}[k]), \quad k \in \mathbb{Z}.$$

Assuming static traffic y , static target buffer size \tilde{b} , and $c[k] = y$, then if we set $\alpha = \frac{1}{T^2}$, we have $b[k+2] = \tilde{b}$. Hence, α is on the order of $1/T^2$, and we update c as

$$c[k+1] = c[k] + \delta \frac{b[k] - \tilde{b}[k]}{T}, \quad k \in \mathbb{Z},$$

where δ is a damping parameter and resides in $(0, 1]$. On

³At steady state, $c_l \neq c_l^{max}$ implies $b_l \leq \tilde{b}_l$. Since $\tilde{b}_l < \underline{b}_l$, we have $b_l < \underline{b}_l$, which implies $p_l = 0$.

one hand, we need small δ to guarantee local stability of TCP/RED/DBA (Theorem 4); on the other hand, we need big δ to avoid long transients. We balance this by using different δ in different buffer regions. We set a threshold $\epsilon > 0$.

If the relative deviation $\left| b/\tilde{b} - 1 \right| \leq \epsilon$, we use a small δ_1 for

local stability. If $\left| b/\tilde{b} - 1 \right| > \epsilon$, we use a big δ_2 to shorten the transient. Typical values are $\epsilon = 0.1$, $\delta_1 = 0.04$, and $\delta_2 = 0.2$. Recall that d , T , ϵ , δ_1 and δ_2 are design parameters, and we summarize the continuous dynamic bandwidth adjustment (continuous DBA) mechanism in Algorithm 1.

Bandwidth levels are far from continuous in practice. Assume that a link has N bandwidth levels

$$c_1 < c_2 < \dots < c_N.$$

To protect throughput, we choose a bandwidth c_k such that $c_k \geq y^{NUM} > c_{k-1}$. Since excess bandwidth $c_k - y^{NUM}$ will probably make $b < \tilde{b}$, we set a hysteresis buffer region $[\underline{b}, \tilde{b}]$ in which we do not reduce c . How to set \underline{b} is beyond the scope of this paper, and we pick $\underline{b} = \tilde{b}/2$ in this paper for simplicity. The discrete dynamic bandwidth adjustment algorithm—given in Algorithm 2—exploits stochasticity to mimic continuous DBA. Recall that d , T , ϵ , and δ_2 are design parameters and we set $\underline{b} = \frac{1}{2}\tilde{b}$.

Algorithm 2 Discrete DBA

At epoch $k \in \mathbb{Z}$, input $c[k-1]$, $b[k-1]$.

1. Integral law.

$$c \leftarrow c[k-1] + \delta_2 \frac{b[k-1] - dc[k-1]}{T};$$

2. Lower and upper bounds.

```

if  $c > c^{max}$  then
   $c \leftarrow c^{max}$ ;
else
  if  $c < c^{min}$  then
     $c \leftarrow c^{min}$ ;
  end if
end if

```

3. Stochastic round off.

```

 $i \leftarrow \operatorname{argmin} \{i \in \{1, \dots, N-1\} : c_i \leq c \leq c_{i+1}\}$ ;
 $p \leftarrow \frac{c_{i+1} - c}{c_{i+1} - c_i}$ ;
 $c \leftarrow \begin{cases} c_i & \text{with probability } p \\ c_{i+1} & \text{with probability } 1 - p \end{cases}$ ;

```

4. Hysteresis.

```

if  $\frac{1}{2}dc[k-1] \leq b[k-1] \leq dc[k-1]$  then
   $c \leftarrow c[k-1]$ ;
end if

```

5. Switching.

```

 $c[k] \leftarrow c$ ;

```

Return $c[k]$ as the link bandwidth during $[kT, (k+1)T)$.

It can be verified that the expectation $E(c)$ does not change during step 3. Besides, c coming from step 1 is equal to the result of continuous DBA when $\left| \frac{b[k-1]}{dc[k-1]} - 1 \right| > \epsilon$. Hence, discrete DBA will hopefully mimic continuous DBA. We will further explore this in section 5.2.

4. ANALYSIS

There are mainly two questions to answer: will traffic y oscillate, and how much throughput reduction will be incurred. Oscillation of y leads to poor quality of service, and might

be induced by the interaction between DBA and TCP/RED. Supporting traffic is the primal goal of the network, hence it would be undesirable if DBA seriously reduce throughput. We answer these two questions in this section.

4.1 Global Stability without Network Delay

Recall that p_l , b_l , x_i and c_l denotes packet drop probability, average buffer size, source rate, and bandwidth respectively. We start with equations (1), (5), (4), and (7) to study the dynamic of TCP/RED/DBA, and call them the *synchronous model* (SM).

Synchronous Model (SM):

$$p_l = \begin{cases} 0 & \text{if } b_l \leq \underline{b}_l \\ \gamma_l(b_l - \underline{b}_l) & \text{if } \underline{b}_l < b_l \leq \bar{b}_l \end{cases} \quad (1)$$

$$\dot{b}_l = \begin{cases} 0 & \text{if } b_l = 0 \text{ \& } y_l < c_l \\ y_l - c_l & \text{otherwise} \end{cases} \quad (5)$$

$$x_i = \begin{cases} x_i^{max} & \text{if } q_i < U_i'(x_i^{max}) \\ U_i'^{-1}(q_i) & \text{if } q_i \geq U_i'(x_i^{max}) \end{cases} \quad (4)$$

$$\dot{c}_l = \begin{cases} 0 & \text{if } c_l = c_l^{min} \text{ \& } b_l < \bar{b}_l \\ & \text{or } c_l = c_l^{max} \text{ \& } b_l > \bar{b}_l \\ \alpha_l (b_l - \bar{b}_l(c_l)) & \text{otherwise} \end{cases} \quad (7)$$

To model RED, we use the more precise equation (1) rather than (2), since the fact that we set $\bar{b}_l < b_l$ can only be captured by (1). We model the dynamics of average buffer size b_l by the dynamics (5) of instantaneous buffer size for simplicity. The actual dynamics of b_l should be a low-pass filtered version of (5). As mentioned in section 2.1, the model (4) we adopt for TCP captures the stationary relationship between source rate and end-to-end packet loss for a generic TCP protocol. To analyze the stability of a specific TCP protocol, we should change to its specific dynamic equation. The dynamic model (7) for c_l is exact for continuous DBA. Note that Model SM does not capture network delay, which we will consider in section 4.3.

According to SM, source rate and link bandwidth converge globally to a unique equilibrium. At the equilibrium, excess bandwidth is completely turned off while source rate is preserved as if DBA has not been implemented. A rigorous statement is given in Theorem 1.

Theorem 1. *Let $(x(t), c(t), b(t), p(t))$ be an arbitrary trajectory generated by SM, then*

$$x(t) \rightarrow x^{NUM}, \quad c(t) \rightarrow \max\{c^{min}, y^{NUM}\}$$

as $t \rightarrow \infty$ provided the following conditions hold:

- C1 $\bar{b}_l(c_l) = d_l > 0$ for $l \in \mathcal{L}$ and $c_l \in C_l$.
- C2 $\bar{b}_l(c_l^{max}) < \bar{b}_l$ for $l \in \mathcal{L}$.
- C3 $U_i''(x_i) \leq -u < 0$, $\forall i \in \mathcal{N}$, $\forall x_i \in \mathcal{X}_i$.

By the design of DBA, Conditions C1 and C2 hold automatically since we set $\bar{b}_l(c_l) = d_l c_l$ with $d_l c_l^{max} < \bar{b}_l$. The utility function U_i is strictly concave so that $U_i''(x_i) < 0$, and Condition C3 holds since \mathcal{X}_i is bounded for each i .

The fact that $x(t) \rightarrow x^{NUM}$ implies no traffic oscillation and zero throughput reduction, and the fact that $c(t) \rightarrow \max\{c^{min}, y^{NUM}\}$ implies zero excess bandwidth at steady state. The proof of Theorem 1 is presented in Appendix A.

4.2 Network Surplus Maximization

The reason DBA reduces c without sacrificing x is that there is a nonempty interval $[0, \bar{b}_l]$ over which p_l is zero. This feature is captured in (1). Without this feature, i.e., $\bar{b}_l = 0$, (1) reduces to (2), which we will use to simplify Model SM. Noting that (2) gives a bijective mapping between b_l and p_l , we can substitute b_l by p_l to reduce the state space dimension. Define $\tilde{p}_l(c_l) := \gamma_l \bar{b}_l(c_l)$ as the packet drop probability corresponding to \bar{b}_l , and call its integral

$$V_l(c_l) := \int_{c_l^{min}}^{c_l} \tilde{p}_l(c) dc$$

the cost function of link l . Note that $V_l(c_l)$ comes from the moving target buffer size $\bar{b}_l(c_l)$ we set, and is strictly convex and increasing. By substituting b_l by p_l , the dynamics (7) for c_l is equivalent to

$$\dot{c}_l = \begin{cases} 0 & \text{if } c_l = c_l^{min} \text{ \& } p_l < V_l'(c_l) \\ & \text{or } c_l = c_l^{max} \text{ \& } p_l > V_l'(c_l) \\ \beta_l (p_l - V_l'(c_l)) & \text{otherwise} \end{cases}, \quad (8)$$

where $\beta_l := \alpha_l / \gamma_l$. Then, Model SM is simplified as below.

Simplified Synchronous Model (SSM):

$$\dot{p}_l = \begin{cases} 0 & \text{if } p_l = 0 \text{ \& } y_l < c_l \\ \gamma_l(y_l - c_l) & \text{otherwise} \end{cases} \quad (6)$$

$$x_i = \begin{cases} x_i^{max} & \text{if } q_i < U_i'(x_i^{max}) \\ U_i'^{-1}(q_i) & \text{if } q_i \geq U_i'(x_i^{max}) \end{cases} \quad (4)$$

$$\dot{c}_l = \begin{cases} 0 & \text{if } c_l = c_l^{min} \text{ \& } p_l < V_l'(c_l) \\ & \text{or } c_l = c_l^{max} \text{ \& } p_l > V_l'(c_l) \\ \beta_l (p_l - V_l'(c_l)) & \text{otherwise} \end{cases} \quad (8)$$

Definition 1. A point (x, c, p) is an equilibrium for SSM, provided that at (x, c, p) , $\dot{p} = 0$, $\dot{c} = 0$ according to (6) and (8), and (x, p) satisfy (4).

Then the steady states of TCP/RED/DBA are equilibrium points for SSM. Similar to NUM for traditional congestion control, we are able to show that steady state of TCP/RED/DBA solves the following *network surplus maximization* (NSM) problem

$$\text{NSM} \begin{cases} \max_{x_i \in \mathcal{X}_i, c_l \in C_l} & \sum_{i \in \mathcal{N}} U_i(x_i) - \sum_{l \in \mathcal{L}} V_l(c_l) \\ \text{s.t.} & y_l \leq c_l, \quad l \in \mathcal{L}. \end{cases}$$

A rigorous statement is given in Theorem 2. Let (x^*, c^*) denote the unique solution to NSM,⁴ and y^* the corresponding traffic throughput. Since V_l is strictly increasing, $c^* = y^*$ if there is no lower bound on c .

Theorem 2. *Equilibrium points of SSM exist, and a point (x, c, p) is an equilibrium point for SSM if and only if (x, c, p) is primal dual optimal for NSM.*

Theorem 2 connects NSM with TCP/RED/DBA, just like the relationship between NUM and TCP/AQM. It implies that we can study the steady-state traffic throughput and link bandwidth of TCP/RED/DBA by solving NSM. The proof of Theorem 2 is based on the duality theory [8, Chapter 5], and omitted due to page limit. Theorem 3 says that TCP/RED/DBA converges globally to its unique steady state (x^*, c^*) .

⁴Since U_i are strictly concave for all i and V_l are strictly convex for all l , solution to NSM is unique.

Theorem 3. Let $(x(t), c(t), p(t))$ be an arbitrary trajectory generated by SSM. Then the source rate $x(t)$ and bandwidth $c(t)$ converge to (x^*, c^*) as $t \rightarrow \infty$ if the following conditions hold:

$$C3 \ U_i''(x_i) \leq -u < 0, \forall i \in \mathcal{N}, \forall x_i \in \mathcal{X}_i.$$

$$C4 \ V_l''(c_l) \geq v > 0, \forall l \in \mathcal{L}, \forall c_l \in \mathcal{C}_l.$$

Condition C3 has already been justified in the discussion after Theorem 1. Condition C4 also holds since $V_l''(c_l) = d_l \gamma_l > 0$ for all l and all c_l . Theorem 3 implies that SSM is globally stable, and solves NSM. The proof of theorem 3 is based on the uniqueness of (x^*, c^*) and contraction-type analysis shown in appendix B. Contraction-type analysis can be found in many references including [3].

4.3 Local Stability with Delay

We proved global convergence to (x^*, c^*) without considering network delay in section 4.2, and will study local stability of (x^*, c^*) under network delay in this section. We add link to source and source to link delays to SSM. For each source link pair $(i, l) \in \mathcal{N} \times \mathcal{L}$, if $l \in \mathcal{L}_i$, let $\tau_{il}(t)$ and $\tau_{li}(t)$ denote the feedforward delay from source i to link l and the feedback delay from link l to source i at time t respectively; if $l \notin \mathcal{L}_i$, define $\tau_{li}(t) = \tau_{il}(t) = 0$ for all t . Then link $l \in \mathcal{L}$ in fact observes traffic

$$\hat{y}_l(t) = \sum_{i \in \mathcal{N}_l} x_i(t - \tau_{il}(t)) = \sum_{i \in \mathcal{N}} R_{li} x_i(t - \tau_{il}(t)), \quad (9)$$

and updates its drop probability according to

$$\dot{p}_l = \begin{cases} 0 & \text{if } p_l = 0 \ \& \ \hat{y}_l < c_l \\ \gamma_l(\hat{y}_l - c_l) & \text{otherwise} \end{cases}. \quad (10)$$

Source $i \in \mathcal{N}$ in fact observes congestion level

$$\hat{q}_i(t) = \sum_{l \in \mathcal{L}_i} p_l(t - \tau_{li}(t)) = \sum_{l \in \mathcal{L}} R_{li} p_l(t - \tau_{li}(t)), \quad (11)$$

and updates its rate as

$$x_i = \begin{cases} x_i^{max} & \text{if } \hat{q}_i < U_i'(x_i^{max}) \\ U_i'^{-1}(\hat{q}_i) & \text{if } \hat{q}_i \geq U_i'(x_i^{max}) \end{cases}. \quad (12)$$

We do not model the delay from a link to itself since it is much shorter, and come to the following simplified asynchronous model (SAM).

Simplified Asynchronous Model (SAM):

$$\dot{p}_l = \begin{cases} 0 & \text{if } p_l = 0 \ \& \ \hat{y}_l < c_l \\ \gamma_l(\hat{y}_l - c_l) & \text{otherwise} \end{cases} \quad (10)$$

$$x_i = \begin{cases} x_i^{max} & \text{if } \hat{q}_i < U_i'(x_i^{max}) \\ U_i'^{-1}(\hat{q}_i) & \text{if } \hat{q}_i \geq U_i'(x_i^{max}) \end{cases} \quad (12)$$

$$\dot{c}_l = \begin{cases} 0 & \text{if } c_l = c_l^{min} \ \& \ p_l < \tilde{p}_l \\ & \text{or } c_l = c_l^{max} \ \& \ p_l > \tilde{p}_l \\ \beta_l(p_l - \tilde{p}_l(c_l)) & \text{otherwise} \end{cases} \quad (8)$$

Definition 2. A point (x, c, p) is an equilibrium for SAM, provided that at (x, c, p) , $\dot{p} = 0$, $\dot{c} = 0$ according to (10) and (8), and (x, p) satisfy (12).

It is not difficult to show that SSM and SAM have the same equilibrium points. Hence, Theorem 2 holds for SAM as well as SSM. Let (x^e, c^e, p^e) be an equilibrium point for SAM, then $(x^e, c^e) = (x^*, c^*)$ —the solution to NSM—according to Theorem 2. We study local stability near (x^*, c^*) through

linearization. Neglect the dynamics of $\tau_{il}(t)$ and $\tau_{li}(t)$ when linearizing (9) and (11) around (x^e, c^e, p^e) as in [5], i.e., treat them as static, and denote the respective values by τ_{il} and τ_{li} . For each source $i \in \mathcal{N}$, its round-trip time τ_i is equal to the sum of source to link delay τ_{il} and link to source delay τ_{li} , for all link l on its path, i.e.,

$$\tau_{il} + \tau_{li} = \tau_i, \quad i \in \mathcal{N}, l \in \mathcal{L}_i.$$

Then, we are able to show that (x^*, c^*) is locally stable, if round-trip times are bounded and link price dynamics and link bandwidth dynamics are not too fast.

Theorem 4. (x^*, c^*) is locally asymptotically stable for SAM, provided the following conditions hold:

$$C3 \ U_i''(x_i) \leq -u < 0, \forall i \in \mathcal{N}, \forall x_i \in \mathcal{X}_i.$$

$$C4 \ V_l''(c_l) \geq v > 0, \forall l \in \mathcal{L}, \forall c_l \in \mathcal{C}_l.$$

$$C5 \ \tau_i \leq \tau \text{ for all } i \in \mathcal{N}.$$

$$C6 \ \gamma_l < \frac{u}{N L \tau} \text{ for each } l \in \mathcal{L}.$$

$$C7 \ \beta_l < \frac{\pi(\pi-2)}{4\tau^2\gamma_l} \text{ for each } l \in \mathcal{L}.$$

Conditions C3 and C4 have been justified in the discussions after Theorem 1 and Theorem 3. Condition C5 holds since network delay is typically below several hundred milliseconds in real networks. β and γ are design parameters we choose. A smaller γ leads to a slower dynamic in p , and a smaller β leads to a slower dynamic in c . By damping the dynamics, Conditions C6 and C7 can be satisfied. Hence, Theorem 4 implies that we can slow down link dynamics to guarantee local stability of (x^*, c^*) . Theorem 4 also implies that the larger the network delay τ , the more likely instability occurs, since the upper bounds for β and γ are inversely proportional to τ . The proof of Theorem 4 is based on the generalized Nyquist theorem [4], and shown in Appendix C.

5. SIMULATION

We evaluate the performance of the dynamic bandwidth adjustment (DBA) mechanisms using the network topology in Figure 4. There are 40 TCP Reno sources connected to a shared link through 1Mb/s access links. To investigate how DBA reacts to traffic changes, we assume that 20 of the TCP sources are long-lived, while the other 20 TCP sources are active only from 200s to 250s. The number of TCP connections change abruptly at 200s and 250s, making bandwidth adjustment more challenging. Since the analysis in section 4 only deals with TCP traffic, we also investigate the impact of non-TCP traffic by introducing 20Mb/s UDP constant-bit-rate traffic in the period 350 ~ 400s.

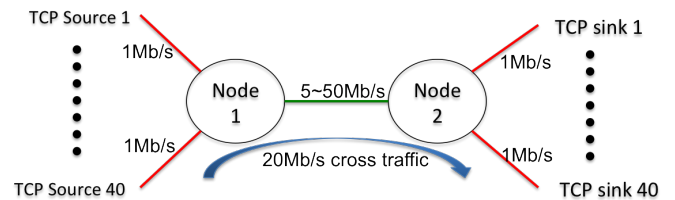


Figure 4: There are 20 long-lived TCP connections and 20 short-lived TCP connections, which start at 200s and end at 250s. There is also 20Mb/s constant-bit-rate UDP cross traffic from 350s to 400s.

We compare the following settings for the shared link.

1. The shared link has static bandwidth of 50Mb/s.
2. The shared link has dynamic bandwidth with continuous levels ranging from 5Mb/s to 50Mb/s. Continuous

DBA, with parameters $T = 100\text{ms}$, $d = 1\text{ms}$, $\epsilon = 0.1$, $\delta_1 = 0.04$, and $\delta_2 = 0.2$, is implemented to adjust the bandwidth.

- The shared link has dynamic bandwidth with discrete levels 5Mb/s, 10Mb/s, ..., 50Mb/s. Discrete DBA, with the same parameters as in setting 2, is implemented to adjust the bandwidth.

All the simulations in this paper use ns-allinone-2.34 on Mac OS X Lion 10.7.2.

5.1 Continuous DBA

Figure 5 shows the traffic throughput and link bandwidth on the shared link, in the static bandwidth setting (no DBA) and continuous bandwidth setting (continuous DBA). All TCP sources use TCP/Reno (loss-based), and have 50ms round-trip time. Due to the 1Mb/s access links (red links in Figure 4), each TCP connection takes up at most 1Mb/s bandwidth. Hence, throughput never reaches maximum bandwidth (50Mb/s) on the shared link. The low utilization level of the shared link provides an opportunity to save energy (by decreasing bandwidth) without reducing the throughput, and continuous DBA effectively exploits this opportunity.

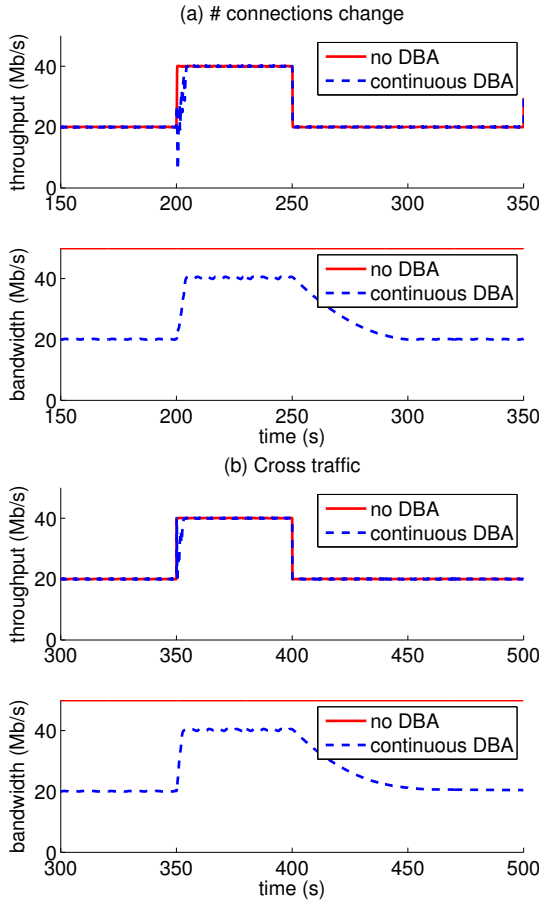


Figure 5: There are 40 TCP connections with 50ms RTT. 20 of the TCP connections are long-lived while the other 20 are active only from 200s to 250s. Additional 20Mb/s UDP constant-bit-rate cross traffic comes at 350s and leaves at 400s.

The following observations can be made from Figure 5.

- At steady state, the throughput is the same for both

settings (no DBA and continuous DBA), even with the presence of non-TCP traffic.

- In the continuous DBA setting, when traffic (abruptly) increases (due to the increase in TCP connections or injection of cross traffic), the bandwidth increases fast. Hence, the effect of DBA is transparent to the TCP sources.
- In the continuous DBA setting, when traffic (abruptly) decreases (due to the decrease in TCP connections or removal of cross traffic), the bandwidth decreases slowly.
- In the continuous DBA setting, link bandwidth equals throughput at steady state.

As we observed, the bandwidth ramps up fast, but ramps down slowly. Fast ramping up makes DBA transparent to the TCP sources, and slow ramping down helps preserving network performance. We now give the reason for the fast ramping up and slow ramping down in DBA. When $c < y$, buffer b can reach its full size b_{full} , and

$$c[k+1] - c[k] \leq \delta_2 \frac{b_{full} - \tilde{b}[k]}{T}.$$

When $c > y$, b can only decrease to 0, and

$$c[k] - c[k+1] \leq \delta_2 \frac{\tilde{b}[k]}{T}.$$

Since $\tilde{b} < b \ll b_{full}$, the ramping down speed has a much smaller upper bound than the ramping up speed. Furthermore, since $\tilde{b} = dc$, the smaller d , the slower c can decrease. When cross traffic goes away at 400s, excess bandwidth clears the buffer to 0 almost instantaneously. Taking average buffer size b to be 0, the dynamics in (7) reduces to

$$\dot{c} = -\delta_2 \frac{\tilde{b}}{T^2} = -\frac{\delta_2 d}{T^2} c.$$

Bandwidth exponentially decays with time constant

$$\frac{T^2}{\delta_2 d} = \frac{(0.1s)^2}{0.2 \times 1ms} = 50s,$$

and it should take

$$50s \times \ln\left(\frac{40}{20}\right) = 34.7s$$

for c to decrease from 40Mbps to 20Mbps, in accordance with the simulation result in Figure 5. We come to the following conclusions for the transients of continuous DBA:

- steady-state is (locally) stable;
- bandwidth ramps up very fast;
- bandwidth ramps down slowly and exponentially, with a time constant inversely proportional to the target queuing delay d .

We provide below two ways to decrease the bandwidth faster to save more energy.

- Increase d . This leads to larger RTT and affects quality of service. Note that $d < b_i/c_i^{max}$, otherwise packet loss kicks in to reduce traffic throughput dramatically.
- Increase δ . However, increasing δ potentially results in instability according to Theorem 4.

5.2 Discrete DBA

Section 5.1 evaluates the performance of continuous DBA, which provides an ‘‘upper bound’’ on what is achievable. In practice, bandwidth levels are discrete, and we may incur performance loss due to this restriction. This motivates us to evaluate discrete DBA in this section.

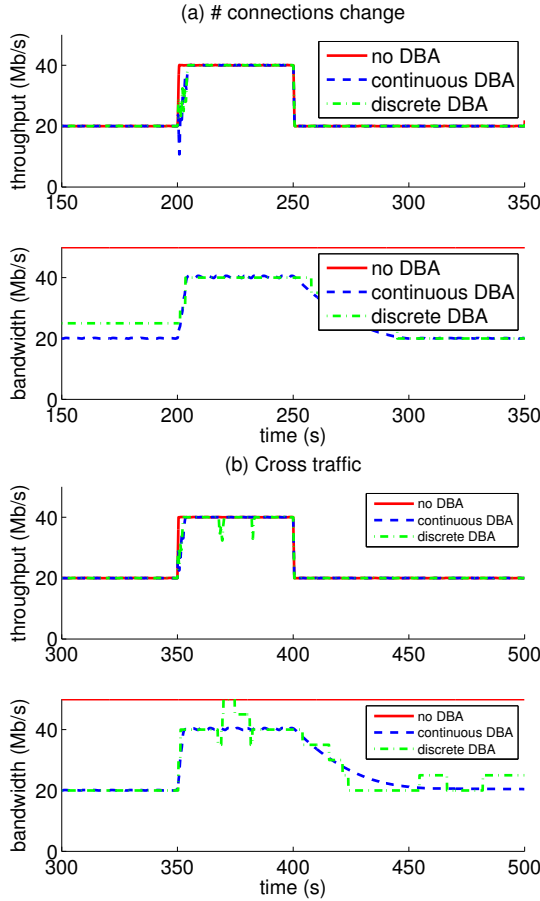


Figure 6: There are 40 TCP connections with 50ms RTT. 20 of the TCP connections are long-lived while the other 20 are active only from 200s to 250s. Additional 20Mb/s UDP constant-bit-rate cross traffic comes at 350s and leaves at 400s.

Figure 6 shows the throughput and bandwidth for all three settings (no DBA, continuous DBA, and discrete DBA) under the same setup as Figure 5. It can be concluded that

- discrete DBA still preserves throughput;
- the bandwidth trajectory generated by discrete DBA roughly follows that of continuous DBA;
- the bandwidth trajectory generated by discrete DBA oscillates around that of continuous DBA—while staying above most of the time—at a low frequency.

Since discrete DBA only has few bandwidth levels to choose from, oscillation around optimal bandwidth is hardly avoidable. Spending most of the time above optimal bandwidth is what we prefer, since traffic protection is of primal concern. In fact, this is achieved by setting hysteric target buffer $[b, \bar{b}]$.

On average, link bandwidth is updated only once every multiple seconds. Frequent link bandwidth updating will cause high overhead that prohibits implementation, and discrete DBA reduces the number of bandwidth updates by exploiting randomness. At each decision epoch, bandwidth is only updated with a small probability. For instance, at 25Mb/s, the bandwidth switches to 20Mb/s with probability 1% (assuming empty buffer). Hence, it takes

$$\frac{1}{0.01}T \approx 10s$$

per update on average. Go on improving the performance of discrete DBA is a topic of our ongoing research.

5.3 Long Network Delay

As network delay increases, stability of flow control becomes more challenging, as reflected in Conditions C6 and C7. This motivates us to evaluate the impact of long network delay.

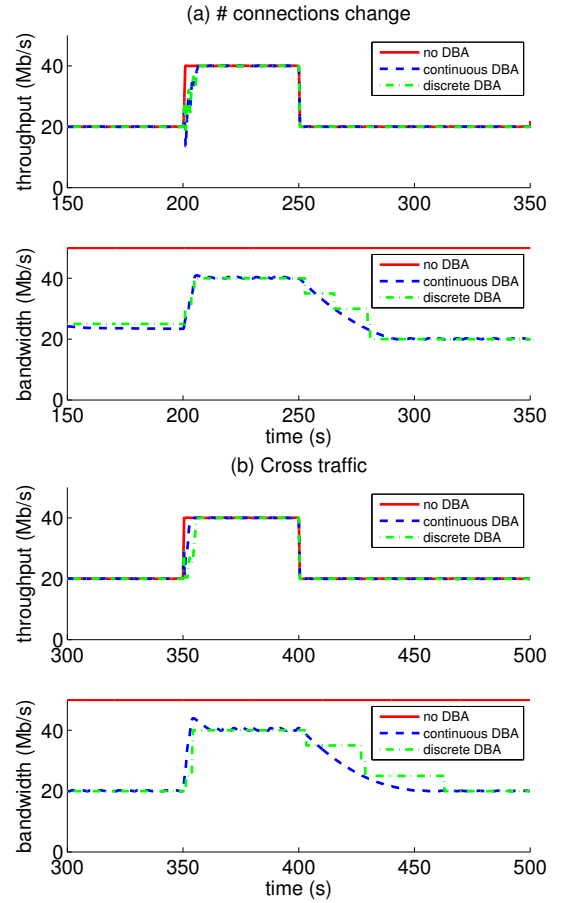


Figure 7: There are 40 TCP connections with 100ms RTT. 20 of the TCP connections are long-lived while the other 20 are active only from 200s to 250s. Additional 20Mb/s UDP constant-bit-rate cross traffic comes at 350s and leaves at 400s.

Figure 7 shows the throughput and bandwidth for all three settings (no DBA, continuous DBA, and discrete DBA) under the same setup as Figure 6, except that the RTT for each flow is 100ms rather than 50ms. It can be seen that the performance of DBA is almost unaffected: the traffic throughput is still as if no DBA is implemented; excess bandwidth is completely turned off at steady state for continuous DBA; bandwidth of discrete DBA follows the bandwidth of continuous DBA.

6. CONCLUSION

We have proposed easily implementable dynamic bandwidth adjustment (DBA) mechanisms that update link bandwidth with local buffer information. We proved global stability of TCP/RED/DBA if network delay is negligible, and local stability if network delay is bounded and link dynamics are sufficiently slow. We also proved that DBA turns off excess bandwidth without reducing throughput, and this is confirmed by ns2 simulations. With a simplified model

for RED, we proved that TCP/RED/DBA solves an optimization problem—which we call the network surplus maximization problem—that is similar to the well-known network utility maximization problem.

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APPENDIX

A. PROOF OF THEOREM 1

Convergence of $\dot{x}(t)$ to 0.

Lemma 1. *For each $i \in \mathcal{N}$, if $U_i''(x_i) \leq -u_i \leq 0$ for all*

x_i in \mathcal{X}_i , then the inequalities

$$|\dot{q}_i(t)| \geq u_i |\dot{x}_i(t)|, \quad \dot{q}_i(t) \dot{x}_i(t) \leq -u_i |\dot{x}_i(t)|^2. \quad (13)$$

hold for all t .

PROOF. $|\dot{q}_i| \geq |U_i''(x_i) \dot{x}_i| = u_i |\dot{x}_i|$, the rest is omitted. \square

Lemma 2. *If condition C3 holds, then the inequality*

$$\sum_l \dot{p}_l(t) \dot{y}_l(t) \leq -u \sum_i \dot{x}_i(t)^2$$

hold for all t .

PROOF.

$$\begin{aligned} \sum_l \dot{p}_l \dot{y}_l &= \sum_l \dot{p}_l \sum_i R_{li} \dot{x}_i = \sum_i \dot{x}_i \sum_l R_{li} \dot{p}_l \\ &= \sum_i \dot{x}_i \dot{q}_i \leq - \sum_i u_i \dot{x}_i^2. \quad \square \end{aligned}$$

Define indicators 1_l as

$$1_l(t) := \begin{cases} 1 & \text{if } b_l(t) > \underline{b}_l \text{ or } b_l(t) = \underline{b}_l \ \& \ \dot{b}_l \geq 0 \\ 0 & \text{otherwise} \end{cases},$$

for $l \in \mathcal{L}$, then $1_l = 0 \Rightarrow (p_l = 0 \ \& \ \dot{p}_l = 0)$ and $1_l = 1 \Rightarrow \dot{p}_l = \gamma_l \dot{b}_l$. We compute

$$\frac{d}{dt} \dot{p}_l^2 = 2\dot{p}_l \ddot{p}_l = 2\dot{p}_l \gamma_l \ddot{b}_l = 2\gamma_l \dot{p}_l (\dot{y}_l - \dot{c}_l),$$

$$\begin{aligned} 1_l \alpha_l \gamma_l \frac{d}{dt} (b_l - \tilde{b}_l)^2 &= 2\alpha_l \gamma_l (b_l - \tilde{b}_l) (\dot{b}_l - \tilde{b}_l'(c_l) \dot{c}_l) 1_l \\ &= 2\alpha_l \gamma_l \dot{b}_l 1_l (b_l - \tilde{b}_l) - 2d_l \gamma_l \dot{c}_l^2 1_l \\ &= 2\alpha_l \dot{p}_l (b_l - \tilde{b}_l) - 2d_l \gamma_l \dot{c}_l^2 1_l. \end{aligned}$$

Then we have

$$\begin{aligned} &\frac{1}{\gamma_l} \frac{d}{dt} \dot{p}_l^2 + 1_l \alpha_l \gamma_l \frac{d}{dt} (b_l - \tilde{b}_l)^2 \\ &= 2\dot{p}_l (\dot{y}_l - \dot{c}_l) + 2\alpha_l (b_l - \tilde{b}_l) \dot{p}_l - 2d_l \gamma_l \dot{c}_l^2 1_l \\ &= 2\dot{p}_l \dot{y}_l + 2\dot{p}_l (\alpha_l (b_l - \tilde{b}_l) - \dot{c}_l) - 2d_l \gamma_l \dot{c}_l^2 1_l. \end{aligned}$$

For each link l , let

$$(t_{2k-1}^l, t_{2k}^l) \subseteq [0, \infty), k \geq 1$$

denote the time segments on which $1_l = 1$ (t_k^l can be ∞). For convenience, we assume that $b_l(0) = \underline{b}_l$ for all $l \in \mathcal{L}$ in the rest of the proof. Extension to other initial points of b is similar. We ignore the superscript l when there is no confusion. Then 1) $t_1 = 0$; 2) $b_l = \underline{b}_l$ at t_k if $t_k \neq \infty$; 3) $b_l \geq \underline{b}_l > \tilde{b}_l(c_l)$ on (t_{2k-1}, t_{2k}) and 4) $\dot{c}_l \geq 0$ on (t_{2k-1}, t_{2k}) . Note that on (t_{2k-1}, t_{2k}) , $\dot{c}_l = 0$ if and only if $c_l = c_l^{max}$, and $\dot{c}_l > 0$ otherwise. If c_l reaches c_l^{max} on (t_{2k-1}, t_{2k}) , define s_{2k} as the first time it attains c_l^{max} . Then we have 1) $c_l = c_l^{max}$ on (s_{2k}, t_{2k}) , 2) $\dot{c}_l = \alpha_l (b_l - \tilde{b}_l)$ on (t_{2k-1}, s_{2k}) , and

$$\begin{aligned} &\int_{t_{2k-1}}^{t_{2k}} 2\dot{p}_l (\alpha_l (b_l - \tilde{b}_l) - \dot{c}_l) dt \\ &= \int_{s_{2k}}^{t_{2k}} 2\gamma_l \dot{b}_l \alpha_l (b_l - \tilde{b}_l(c_l^{max})) dt \\ &= \alpha_l \gamma_l (b_l^2(t_{2k}) - b_l^2(s_{2k})) - 2\alpha_l \gamma_l \tilde{b}_l(c_l^{max}) (b_l(t_{2k}) - b_l(s_{2k})) \\ &= \alpha_l \gamma_l (\underline{b}_l - b_l(s_{2k})) (\underline{b}_l + b_l(s_{2k}) - 2\tilde{b}_l(c_l^{max})) \leq 0. \end{aligned}$$

If c_l does not reach c_l^{max} on (t_{2k-1}, t_{2k}) , the integration above is obviously 0 since $\dot{c}_l = \alpha_l(b_l - \tilde{b}_l)$ on (t_{2k-1}, t_{2k}) . Hence,

$$\begin{aligned} & \int_0^\infty 2\dot{p}_l(\alpha_l(b_l - \tilde{b}_l) - \dot{c}_l) dt \\ &= \sum_{k \geq 1} \int_{t_{2k-1}}^{t_{2k}} 2\dot{p}_l(\alpha_l(b_l - \tilde{b}_l) - \dot{c}_l) dt \leq 0. \end{aligned}$$

We also have

$$\begin{aligned} & \int_{t_{2k-1}}^{t_{2k}} (-2d_l \gamma_l 1_l \dot{c}_l^2) dt \\ &= -2d_l \alpha_l \gamma_l \int_{t_{2k-1}}^{t_{2k}} \dot{c}_l(b_l - \tilde{b}_l) dt \\ &\leq -2d_l \alpha_l \gamma_l \int_{t_{2k-1}}^{t_{2k}} \dot{c}_l(b_l - \tilde{b}_l) dt \\ &= d_l \alpha_l \gamma_l (d_l c_l^2 - 2b_l c_l) \Big|_{t_{2k-1}}^{t_{2k}} \\ &= \alpha_l \gamma_l (d_l c_l - b_l)^2 \Big|_{t_{2k-1}}^{t_{2k}}. \end{aligned}$$

Hence,

$$\int_0^\infty (-2d_l \gamma_l 1_l \dot{c}_l^2) dt \leq \alpha_l \gamma_l \sum_{k \geq 1} (d_l c_l - b_l)^2 \Big|_{t_{2k-1}}^{t_{2k}}.$$

We have

$$\begin{aligned} & \int_0^\infty \frac{d}{dt} \left(\frac{1}{\gamma_l} \dot{p}_l^2 + \alpha_l \gamma_l (b_l - \tilde{b}_l)^2 \right) dt \\ &= \alpha_l \gamma_l \sum_{k \geq 1} (b_l - \tilde{b}_l)^2 \Big|_{t_{2k}}^{t_{2k+1}} \\ &+ \int_0^\infty \left(2\dot{p}_l \dot{y}_l + 2\dot{p}_l(\alpha_l(b_l - \tilde{b}_l) - \dot{c}_l) - 2d_l \gamma_l \dot{c}_l^2 1_l \right) dt \\ &\leq \alpha_l \gamma_l \sum_{k \geq 1} (b_l - \tilde{b}_l)^2 \Big|_{t_{2k}}^{t_{2k+1}} + 2 \int_0^\infty \dot{p}_l \dot{y}_l dt \\ &+ \alpha_l \gamma_l \sum_{k \geq 1} (d_l c_l - b_l)^2 \Big|_{t_{2k-1}}^{t_{2k}} \\ &= 2 \int_0^\infty \dot{p}_l \dot{y}_l dt - \alpha_l \gamma_l (\tilde{b}_l(0) - b_l)^2. \end{aligned}$$

Sum up the previous inequality for all $l \in \mathcal{L}$ yields

$$\begin{aligned} & \int_0^\infty \frac{d}{dt} \sum_l \left(\frac{1}{\gamma_l} \dot{p}_l^2 + \alpha_l \gamma_l (b_l - \tilde{b}_l)^2 \right) dt \\ &\leq 2 \int_0^\infty \sum_l \dot{p}_l \dot{y}_l dt - \sum_l \alpha_l \gamma_l (b_l - \tilde{b}_l(0))^2 \\ &\leq - \sum_l \alpha_l \gamma_l (b_l - \tilde{b}_l(0))^2 - 2u \sum_i \int_0^\infty \dot{x}_i^2 dt. \end{aligned}$$

Since the left hand side is finite,

$$\int_0^\infty \dot{x}_i^2 dt$$

is finite for all i . Since \dot{x}_i is Lipschitz, $\dot{x}_i \rightarrow 0$ as $t \rightarrow \infty$.

Convergence of $x(t)$ to x^{NUM} .

Define the set

$$\mathcal{S} := \{(x, c, p, b) \mid \dot{x} = 0 \text{ according to Model SM}\},$$

and \mathcal{M} its largest invariant set. Then starting from any element $(x, c, p, b) \in \mathcal{M}$, x and y are time-invariant. If

$y_l > c_l^{max}$ for some $l \in \mathcal{L}$, then $b_l(t)$ blows up, driving x_i to 0 for $i \in \mathcal{N}_l$, contradict with $\dot{x} = 0$. Hence, $y_l \leq c_l^{max}$ for $l \in \mathcal{L}$ on \mathcal{M} . Each link l has a state (c_l, b_l) and constant control input y_l . From the phase plot in Figure 3, it is not difficult to show that (c_l, b_l) will converge to some equilibrium point (c_l^e, b_l^e) . Let p_l^e be the packet drop probability corresponding to b_l^e . By checking the KKT conditions, (x, p^e) is primal dual optimal for Problem NUM with $c_l = c_l^{max}$ for all $l \in \mathcal{L}$. Hence, $x = x^{NUM}$. Now we have shown that $x \in \mathcal{M} \Rightarrow x = x^{NUM}$. Similar to Lasalle's theorem, we can show that $(x(t), c(t), p(t), b(t)) \rightarrow \mathcal{M}$. Consequently, $x(t) \rightarrow x^{NUM}$ as $t \rightarrow \infty$.

Convergence of $c(t)$ to $\max\{c^{min}, y^{NUM}\}$.

Consider each link l as an autonomous system with state (c_l, b_l) and input y_l . According to the phase plot in Figure 3, the system of link l is globally asymptotically stable with constant input y_l .⁵ Since $x(t) \rightarrow x^{NUM}$, $y_l \rightarrow y_l^{NUM}$ for all l . If the input is y_l^{NUM} , the unique equilibrium point (c_l^e, b_l^e) satisfies that $c_l^e = \max\{c_l^{min}, y_l^{NUM}\}$. Consequently, $c_l(t) \rightarrow \max\{c_l^{min}, y_l^{NUM}\}$ as $t \rightarrow \infty$ for all $l \in \mathcal{L}$.

This completes the proof of Theorem 1.

B. PROOF OF THEOREM 3

Let (x^*, c^*, p^*) be an equilibrium for Model SSM, $(x(t), c(t), p(t))$ be an arbitrary trajectory generated by SSM. Define $\delta x(t) := x(t) - x^*$, $\delta c(t) := c(t) - c^*$, $\delta p(t) := p(t) - p^*$ as the deviation of the trajectory from the equilibrium.

Lemma 3. For each $i \in \mathcal{N}$, if

$$U_i''(x_i) \leq -u_i \leq 0, \quad \forall x_i \in \mathcal{X}_i,$$

then

$$|\delta q_i(t)| \geq u_i |\delta x_i(t)|, \quad \delta q_i(t) \delta x_i(t) \leq -u_i (\delta x_i(t))^2 \quad (14)$$

for all t .

PROOF. For brevity, we abbreviate the argument t . We only show proof for the case $\delta x_i \leq 0$. Proof for the case $\delta x_i \geq 0$ is similar. If $\delta x_i = 0$, (14) is satisfied. Otherwise,

$$x_i^{min} \leq x_i < x_i^* \leq x_i^{max}.$$

It follows that

$$q_i \geq U_i'(x_i) > U_i'(x_i^*) \geq q_i^*,$$

and

$$\begin{aligned} \delta q_i &\geq U_i'(x_i) - U_i'(x_i^*) = \int_{x_i^*}^{x_i} -U_i''(x) dx \\ &\geq \int_{x_i}^{x_i^*} u_i dx = -u_i \delta x_i. \end{aligned}$$

It follows that (14) is satisfied. \square

Define $B := \text{diag}(\beta_1, \dots, \beta_L)$, $\Gamma := \text{diag}(\gamma_1, \dots, \gamma_L)$. Since $\gamma_l > 0$, $\beta_l > 0$ for all $l \in \mathcal{L}$, diagonal matrices B and Γ are positive definite.

Lemma 4. For each $l \in \mathcal{L}$,

$$\frac{d}{dt} \left(\frac{(p_l - p_l^*)^2}{\gamma_l} \right) \leq 2(p_l - p_l^*)(y_l - c_l) \quad (15)$$

$$\leq 2\delta p_l(\delta y_l - \delta c_l). \quad (16)$$

⁵When the input y_l is time-invariant, the dynamic of link l can be shown in a 2-dimensional phase plot, from which its global asymptotical stability is obvious.

PROOF. For each $l \in \mathcal{L}$, define

$$[a]_0 := \begin{cases} 0 & \text{if } p_l = 0 \text{ \& } a < 0 \\ a & \text{otherwise} \end{cases},$$

then

$$\begin{aligned} \frac{d}{dt} \left(\frac{(p_l - p_l^*)^2}{\gamma_l} \right) &= \frac{2}{\gamma_l} (p_l - p_l^*) \dot{p}_l \\ &= \frac{2}{\gamma_l} (p_l - p_l^*) [\gamma_l (y_l - c_l)]_0 \\ &= 2(p_l - p_l^*) [y_l - c_l]_0 \end{aligned}$$

If $[y_l - c_l]_0 = y_l - c_l$, (15) is satisfied with equality. Otherwise, $[y_l - c_l]_0 = 0$, $p_l = 0$ and $y_l < c_l$. It follows that the left hand side of (15) is zero, and the right hand side of (15) is non-negative. Hence, (15) is always satisfied. Besides,

$$\begin{aligned} &2(p_l - p_l^*)(y_l - c_l) \\ &= 2\delta p_l (y_l - y_l^* + c_l^* - c_l) + 2\delta p_l (y_l^* - c_l^*) \\ &= 2\delta p_l (\delta y_l - \delta c_l) + 2p_l (y_l^* - c_l^*) - 2p_l^* (y_l^* - c_l^*) \\ &\leq 2\delta p_l (\delta y_l - \delta c_l). \end{aligned}$$

The last inequality holds since $p_l^* (y_l^* - c_l^*) = 0$, $p_l \geq 0$, and $y_l^* \leq c_l^*$. \square

Lemma 5. For each $l \in \mathcal{L}$, if

$$V_l''(c_l) \geq v_l \geq 0, \quad \forall c_l \in \mathcal{C}_l,$$

then

$$\frac{d}{dt} \left(\frac{(c_l - c_l^*)^2}{\beta_l} \right) \leq 2(c_l - c_l^*)(p_l - V_l'(c_l)) \quad (17)$$

$$\leq 2\delta c_l \delta p_l - 2v_l (\delta c_l)^2. \quad (18)$$

PROOF. For each $l \in \mathcal{L}$, define

$$[a]_{c_l^{min}}^{c_l^{max}} := \begin{cases} 0 & \text{if } c_l = c_l^{min} \text{ \& } a < 0 \\ & \text{or } c_l = c_l^{max} \text{ \& } a > 0 \\ a & \text{otherwise} \end{cases},$$

then,

$$\begin{aligned} \frac{d}{dt} \left(\frac{(c_l - c_l^*)^2}{\beta_l} \right) &= \frac{2}{\beta_l} (c_l - c_l^*) \dot{c}_l \\ &= \frac{2}{\beta_l} (c_l - c_l^*) [\beta_l (p_l - V_l'(c_l))]_{c_l^{min}}^{c_l^{max}} \\ &= 2(c_l - c_l^*) [p_l - V_l'(c_l)]_{c_l^{min}}^{c_l^{max}} \end{aligned}$$

If $[p_l - V_l'(c_l)]_{c_l^{min}}^{c_l^{max}} = p_l - V_l'(c_l)$, (17) is satisfied with equality. Otherwise, $[p_l - V_l'(c_l)]_{c_l^{min}}^{c_l^{max}} = 0$, the left hand side of (17) is zero. Besides, either $p_l > V_l'(c_l)$, $c_l = c_l^{max}$ or $p_l < V_l'(c_l)$, $c_l = c_l^{min}$. For both cases, the right hand side of (17) is non-negative. Hence, (17) is always satisfied. Besides,

$$\begin{aligned} &2(c_l - c_l^*)(p_l - V_l'(c_l)) \\ &= 2\delta c_l (p_l - p_l^* + V_l'(c_l^*) - V_l'(c_l)) + 2\delta c_l (p_l^* - V_l'(c_l^*)) \\ &\leq 2\delta c_l \delta p_l + 2\delta c_l (V_l'(c_l^*) - V_l'(c_l)) \\ &= 2\delta c_l \delta p_l + 2\delta c_l \int_{c_l}^{c_l^*} V_l''(c) dc \\ &\leq 2\delta c_l \delta p_l + 2\delta c_l \int_{c_l}^{c_l^*} v_l dc \\ &= 2\delta c_l \delta p_l - 2v_l (\delta c_l)^2. \end{aligned}$$

Hence, (18) holds. \square

Lemma 6. If Conditions C3 and C4 hold, then

$$\begin{aligned} &\frac{d}{dt} \left[\begin{pmatrix} \delta p \\ \delta c \end{pmatrix}^T \begin{pmatrix} \Gamma^{-1} & 0 \\ 0 & B^{-1} \end{pmatrix} \begin{pmatrix} \delta p \\ \delta c \end{pmatrix} \right] \\ &\leq -2 \sum_{i \in \mathcal{N}} u(\delta x_i)^2 - 2 \sum_{l \in \mathcal{L}} v(\delta c_l)^2. \end{aligned} \quad (19)$$

PROOF. If Conditions C3 and C4 hold, then

$$\begin{aligned} &\frac{d}{dt} \left[\begin{pmatrix} \delta p \\ \delta c \end{pmatrix}^T \begin{pmatrix} \Gamma^{-1} & 0 \\ 0 & B^{-1} \end{pmatrix} \begin{pmatrix} \delta p \\ \delta c \end{pmatrix} \right] \\ &= \sum_{l \in \mathcal{L}} \frac{d}{dt} \left(\frac{(p_l - p_l^*)^2}{\gamma_l} \right) + \sum_{l \in \mathcal{L}} \frac{d}{dt} \left(\frac{(c_l - c_l^*)^2}{\beta_l} \right) \\ &\leq \sum_{l \in \mathcal{L}} 2\delta p_l (\delta y_l - \delta c_l) + \sum_{l \in \mathcal{L}} 2(\delta c_l \delta p_l - v(\delta c_l)^2) \\ &= 2 \sum_{l \in \mathcal{L}} \delta p_l \delta y_l - 2 \sum_{l \in \mathcal{L}} v(\delta c_l)^2 \\ &= 2 \sum_{i \in \mathcal{N}} \delta x_i \delta q_i - 2 \sum_{l \in \mathcal{L}} v(\delta c_l)^2 \\ &\leq -2 \sum_{i \in \mathcal{N}} u(\delta x_i)^2 - 2 \sum_{l \in \mathcal{L}} v(\delta c_l)^2. \end{aligned}$$

The first inequality is due to (16) and (18), and the second inequality is due to (14). \square

Proof of Theorem 3.

For brevity, define $z := (\delta p^T, \delta c^T)^T$, $M := \text{diag}(\Gamma^{-1}, B^{-1})$. Then (19) implies that

$$\begin{aligned} &\int_0^\infty \frac{d}{dt} (z^T M z) dt \\ &\leq -2u \sum_{i \in \mathcal{N}} \int_0^\infty (\delta x_i)^2 dt - 2v \sum_{l \in \mathcal{L}} \int_0^\infty (\delta c_l)^2 dt. \end{aligned}$$

Since the left hand side is bounded and δx_i is Lipschitz, we have $\delta x_i(t) \rightarrow 0$ as $t \rightarrow \infty$ for all $i \in \mathcal{N}$. Similarly, $\delta c_l(t) \rightarrow 0$ as $t \rightarrow \infty$ for all $l \in \mathcal{L}$. As $\delta x(t)/\delta c(t)$ denotes the deviation from $x(t)/c(t)$ to x^*/c^* , source rate $x(t)$ and link bandwidth $c(t)$ converge to the unique solution (x^*, c^*) to NSM. \square

C. PROOF OF THEOREM 4

Let (x^*, c^*, p^*) be an equilibrium for Model SAM, then it's also an equilibrium for Model SSM, and primal dual optimal point for Problem NSM. Let $(x(t), c(t), p(t))$ be an arbitrary trajectory generated by SAM, and define $\delta x(t) := x(t) - x^*$, $\delta c(t) := c(t) - c^*$, $\delta p(t) := p(t) - p^*$ as the deviation from the trajectory to the equilibrium point (x^*, c^*, p^*) .

We use the generalized Nyquist theorem to show local stability of (x^*, c^*) . We first linearize dynamic equations (8,10,12) near (x^*, c^*, p^*) , and then derive the loop transfer matrix. In the end, we evaluate the eigenvalues of the loop transfer matrix and show that they will not encircle the -1 point as frequency goes from $-\infty$ to ∞ , if the conditions in Theorem 4 holds.

Linearization.

For each link $l \in \mathcal{L}$, c_l is restricted to the region $[c_l^{min}, c_l^{max}]$. If the constraint is active, c_l will be fixed at either c_l^{min} or c_l^{max} . Effectively, it is equivalent to c_l being static, and we only need to linearize (8) for the links that c_l is not static. For the same reason, we only linearize (10) and (12) for the links and sources that p_l and x_i are not effectively static. Without

loss of generality, we assume that all links and sources are not effectively static, leading us to

$$\dot{c}_l(t) = \beta_l(\delta p_l(t) - v_l \delta c_l(t)), \quad l \in \mathcal{L}, \quad (20)$$

$$\dot{\delta p}_l(t) = \gamma_l \left(\sum_{i \in \mathcal{N}} R_{li} \delta x_i(t - \tau_{il}) - \delta c_l(t) \right), \quad l \in \mathcal{L}, \quad (21)$$

$$\delta x_i(t) = -u_i \sum_{l \in \mathcal{L}} R_{li} \delta p_l(t - \tau_{il}), \quad i \in \mathcal{N}, \quad (22)$$

where $v_l := V_l''(c_l^*)$ for $l \in \mathcal{L}$ and $u_i := -1/U_i''(x_i^*)$ for $i \in \mathcal{N}$.

Loop Transfer Matrix.

Let $R(s)$ denote the $L \times N$ matrix whose (l, i) entry is given by $R_{li}(s) := R_{li} \exp(-s\tau_{il})$. Let $X(s)$, $P(s)$, $C(s)$ denote the Laplacian transformation of $\delta x(t)$, $\delta p(t)$, and $\delta c(t)$ respectively. Define $U := \text{diag}(u_1, \dots, u_N)$, $V := \text{diag}(v_1, \dots, v_L)$. Equation (20) implies

$$C(s) = \text{diag} \left(\frac{\beta_l}{s + \beta_l v_l} \right) P(s).$$

Equation (21) implies

$$sP(s) = \Gamma R(s)X(s) - \Gamma C(s).$$

Equation (22) implies

$$\begin{aligned} X_i(s) &= -u_i \sum_{l \in \mathcal{L}} R_{li} P_l(s) \exp(-s\tau_{li}) \\ &= -u_i \exp(-s\tau_i) \sum_{l \in \mathcal{L}} R_{li} P_l(s) \exp(s\tau_{il}) \end{aligned}$$

for $i \in \mathcal{N}$. Hence,

$$X(s) = -\text{diag} (u_i e^{-s\tau_i}) R^T(-s)P(s).$$

It is not difficult to show that the loop transfer matrix is

$$L(s) = \frac{1}{s} \Gamma R(s) \text{diag} (u_i e^{-s\tau_i}) R^T(-s) + \text{diag} \left(\frac{\beta_l \gamma_l}{s(s + \beta_l v_l)} \right).$$

Eigenvalues.

Define matrices

$$A := \text{diag} \left(\frac{u_i e^{-jw\tau_i}}{jw} \right), \quad D := \text{diag} \left(\frac{\beta_l \gamma_l}{jw(jw + \beta_l v_l)} \right).$$

Let λ denote an arbitrary eigenvalue of $L(jw)$, and z its corresponding eigenvector. To this end, abbreviate $R(jw)$ as R , and use the superscript H to denote conjugate transpose, then

$$\begin{aligned} \lambda z &= (\Gamma R A R^H + D)z, \\ \lambda z^H \Gamma^{-1} z &= z^H (R A R^H + \Gamma^{-1} D)z, \\ \lambda &= \frac{z^H R A R^H z}{z^H \Gamma^{-1} z} + \frac{z^H \Gamma^{-1} D z}{z^H \Gamma^{-1} z}. \end{aligned}$$

Define

$$\lambda_1 := \frac{z^H R A R^H z}{z^H \Gamma^{-1} z}, \quad \lambda_2 := \frac{z^H \Gamma^{-1} D z}{z^H \Gamma^{-1} z},$$

then $\lambda = \lambda_1 + \lambda_2$. Denote $\xi := R^H z$, then

$$\|\xi\|^2 = z^H R R^H z \leq L N \|z\|^2.$$

Condition C3 implies that $u_i \leq 1/u$. Define $\gamma := \max_{i \in \mathcal{L}} \{\gamma_i\}$, $f_i := \frac{e^{-jw\tau_i}}{jw\tau_i}$, then

$$\begin{aligned} \lambda_1 &= \frac{\xi^H A \xi}{z^H \Gamma^{-1} z} \\ &\in [0, 1] * \gamma \frac{\xi^H A \xi}{z^H z} \\ &\subseteq [0, 1] * \gamma L N \frac{\xi^H A \xi}{\xi^H \xi} \\ &= [0, \gamma L N] * \frac{\sum_{i \in \mathcal{N}} u_i \tau_i f_i \|\xi_i\|^2}{\sum_{i \in \mathcal{N}} \|\xi_i\|^2} \\ &= \left[0, \frac{\gamma L N \tau}{u} \right] * \sum_{i \in \mathcal{N}} \frac{u u_i \tau_i \|\xi_i\|^2}{\sum_{j \in \mathcal{N}} \tau \|\xi_j\|^2} f_i \\ &\subseteq [0, 1] * \sum_{i \in \mathcal{N}} \frac{u u_i \tau_i \|\xi_i\|^2}{\sum_{j \in \mathcal{N}} \tau \|\xi_j\|^2} f_i. \end{aligned}$$

Since $\tau_i \leq \tau$, $u u_i \leq 1$ for all $i \in \mathcal{N}$, λ_1 lies in the convex hull of $\{0, f_1, \dots, f_N\}$. Define $d_l := \frac{\beta_l \gamma_l}{jw(jw + \beta_l v_l)}$ for all $l \in \mathcal{L}$, then d_l lies in the third orthant for all $w > 0$, and

$$\lambda_2 = \sum_{l \in \mathcal{L}} \frac{\|z_l\|^2 / \gamma_l}{\sum_{k \in \mathcal{L}} \|z_k\|^2 / \gamma_k} d_l$$

lies in the convex hull of $\{d_1, \dots, d_L\}$.

Proof of Theorem 4.

When $0 < w < \frac{\pi}{2\tau}$, f_i is in the third orthant for all $i \in \mathcal{N}$. Hence, λ_1 is in the third orthant. Since d_l is also in the third orthant for all $l \in \mathcal{L}$ and all $w > 0$, λ_2 is in the third orthant. Hence, $\lambda = \lambda_1 + \lambda_2$ is in the third orthant when $0 < w < \frac{\pi}{2\tau}$.

When $w \geq \frac{\pi}{2\tau}$, Condition C7 implies that

$$|d_l| \leq \frac{\beta_l \gamma_l}{\frac{\pi}{2\tau} \sqrt{\left(\frac{\pi}{2\tau}\right)^2 + (\beta_l v_l)^2}} \leq \frac{4\tau^2}{\pi^2} \beta_l \gamma_l < 1 - \frac{2}{\pi}.$$

Hence, $|\lambda_2| < 1 - \frac{2}{\pi}$. Note that λ_1 lies in the shaded area in Figure 8, and λ_2 lies in the third orthant. $|\lambda_2| \geq 1 - \frac{2}{\pi}$ is necessary for λ to wind the -1 point. Hence, λ does not wind the -1 point when $w \geq \frac{\pi}{2\tau}$.

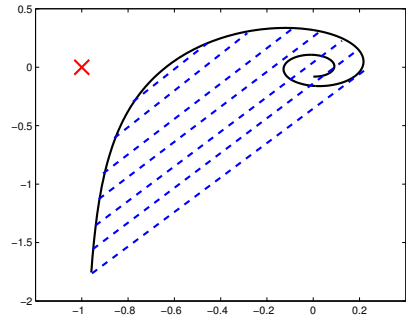


Figure 8: Nyquist plot for $f_i, i \in \mathcal{N}$.

Hence, λ does not wind the -1 point as w goes from 0 to ∞ . Similarly, λ does not cross the -1 point as w goes from $-\infty$ to 0. Since λ is an arbitrary eigenvalue of the loop transfer matrix, the equilibrium point (x^*, c^*, p^*) is locally asymptotically stable by the generalized Nyquist theorem. \square