Characterizing distribution rules for cost sharing games

Ragavendran Gopalakrishnan
Computing and Math. Sciences
California Institute of Tech.

Jason R. Marden
Electrical, Computer and Energy Engineering
University of Colorado at Boulder

Adam Wierman
Computing and Math. Sciences
California Institute of Tech.

***Abstract*—We consider the problem of designing the distribution rule used to share "welfare" (cost or revenue) among individually strategic agents. There are many distribution rules known to guarantee the existence of a (pure Nash) equilibrium in this setting, e.g., the Shapley value and its weighted variants; however a characterization of the space of distribution rules that yield the existence of a Nash equilibrium is unknown. Our work provides a step towards such a characterization. We prove that when the welfare function is strictly submodular, a budget-balanced distribution rule guarantees equilibrium existence for all games (i.e., all possible sets of resources, agent action sets, etc.) if and only if it is a weighted Shapley value.**

I. Introduction

How should the cost incurred (revenue generated) by a set of self-interested agents be shared among them? This fundamental question has led to a large literature in economics over the last decades [24], [20], [28], [21], [19], and more recently in computer science [8], [2], [5], [6], [16]. A classic framework within which to study this question is that of cost sharing games, in which there is a set of "agents" making strategic choices of which "resources" to utilize. Each resource generates a "welfare" (cost or revenue) depending on the set of agents that choose the resource. The focus is on finding *budget-balanced* distribution rules that provide "stable" and/or "fair" allocations, which is traditionally formalized by the concept of the *core* – the set of feasible distribution rules that guarantee a stable grand coalition.

Recently, there is an emerging focus on weaker notions of "stability" and, in particular, a (pure Nash) equilibrium for the agents, which is our focus in this work. This focus is driven by applications such as network-cost sharing [2], [6] where individually strategic behavior is commonly assumed. Additionally, the notion of an equilibrium is natural if cost sharing distribution rules are used to design utilities for distributed agents in the context of a game-theoretic approach to distributed control [16], [9].

Existing literature on cost sharing games provides several distribution rules that guarantee equilibrium existence [24], [23], [27], [26]. Perhaps, the most famous such distribution rule is the Shapley value [24], which guarantees the existence of a Nash equilibrium in any game, and for certain classes of games such as convex games, is always in the core. A generalization of the Shapley value which exhibits the same properties is the weighted Shapley value [23].

Although (weighted) Shapley value distribution rules guarantee equilibrium existence, this is only one of many desirable properties. Perhaps the next most important property is that these equilibria should be "efficient" in the sense of maximizing the social welfare. In order to provide the tools necessary to optimize efficiency (and other properties) while still always ensuring equilibrium existence, researchers have recently sought to provide characterizations of the class of distribution rules that guarantee equilibrium existence. Providing such a characterization is the goal of this paper.

The first step toward this goal is the work of Chen, Roughgarden, & Valiant [6], who prove that the only budget-balanced distribution rules that guarantee equilibrium existence in all cost sharing games are weighted Shapley value distribution rules. Following on [6], Marden & Wierman [17] provide the parallel characterization in the context of revenue sharing games. Though the characterizations in [6] and [17] seem general, they actually provide only a worst-case characterization. In particular, the proofs in [6], [17] consist of exhibiting a specific "worst-case" welfare function which requires that weighted Shapley value distribution rules be used. Thus, the question of characterizing the space of distribution rules for any specific welfare function remains open. In practice, it is exactly this issue that is important: when designing a distribution rule, one *knows* the specific welfare function for the situation. In such a situation, there may be distribution rules other than weighted Shapley value rules that also guarantee the existence of an equilibrium. In particular, recent work has shown that such settings do exist [16].

In this paper, we seek to provide a more detailed characterization of the space of distribution rules by understanding, *for specific welfare functions*, what rules guarantee the existence of a (pure Nash) equilibrium. In particular, our focus is on the class of submodular welfare functions. Submodular welfare functions are quite common and are found, for example, in power control and coverage problems in sensor networks [4], [16], wireless access point assignment and frequency selection [11], and influence maximization [12].

Our main result (Theorem 2) states that, given any strictly submodular welfare function, only weighted Shapley value distribution rules guarantee equilibrium existence in all games. Thus, we show, perhaps surprisingly, that the result of [6] holds much more generally. In particular, we show that it is not the existence of some worst-case welfare function which limits the design of "desirable" distribution rules to weighted Shapley values. In fact, even under practical (submodular) welfare functions, the design of distribution rules is constrained to the weighted Shapley value. This characterization means that in many practical settings, it is possible to optimize other desirable properties (such as the "efficiency" of the equilibrium) within the class of weighted Shapley value distribution rules.

II. Model

In this work, we consider a simple, but general, model of a "welfare" (cost or revenue) sharing game, where there is a set of self-interested agents/players $N = \{1, \ldots, n\}$ who each choose from a set $R = \{r_1, \ldots, r_m\}$, the resources to which to allocate themselves. Each agent $i \in N$ is capable of selecting potentially multiple resources in R; therefore, we say that agent i has action set $\mathcal{A}_i \subseteq 2^R$. The resulting action profile, or (joint) allocation, is a tuple $a = (a_1, \ldots, a_n) \in \mathcal{A}$ where the set of all possible allocations is denoted by $\mathcal{A} = \mathcal{A}_1 \times \ldots \times \mathcal{A}_n$. We occasionally denote an allocation a by (a_i, a_{-i}) where a_{-i} denotes the allocation of all agents except agent i.

Each allocation generates a welfare, $\mathcal{W}(a)$, which needs to be shared completely among the agents, i.e., the allocation is *budget-balanced*. In this work, we assume $\mathcal{W}(a)$ is *(linearly) separable* and *scalable* across resources, i.e., $\mathcal{W}(a) = \sum_{r \in R} v_r W(\{a\}_r)$ where $\{a\}_r = \{i \in N : r \in a_i\}$ is the set of agents that are allocated to resource r in a, $v_r \in \mathbb{R}_{++}$ is the local scaling factor of resource r, and $W : 2^N \to \mathbb{R}$ is the welfare function that is scaled at each resource. These are standard assumptions, e.g., see [6], [16], and are quite general. Note that this model incorporates both revenue and cost sharing games, since we allow for the welfare function W to be either positive or negative.

The manner in which the welfare is shared among the agents determines the utility function $U_i : \mathcal{A} \to \mathbb{R}$ that agent i seeks to maximize. Because the welfare function is assumed to be separable and scalable, it is natural that the utility functions should follow suit. The motivation for scalability is self-evident. To motivate separability, note that this corresponds to welfare garnered from each resource being distributed among only the agents allocated to that resource, which is most often appropriate, e.g., in profit sharing. This results in $U_i(a) = \sum_{r \in a_i} v_r f(i, \{a\}_r)$, where $f : N \times 2^N \to \mathbb{R}$ is the distribution rule, i.e., $f(i,S)$ is the fraction of the welfare allocated to agent $i \in S$ when sharing with S. Recall that we require f to be budget-balanced, which means that for any player set $S \subseteq N$, $\sum_{i \in S} f(i,S) = W(S)$.

To summarize, we can specify a welfare sharing game G using the tuple $G = \left(N, R, \{\mathcal{A}_i\}_{i \in N}, \{v_r\}_{r \in R}, f, W\right)$, where the design of f is the focus of this paper.

The primary goals when designing f are to guarantee (i) equilibrium existence, and (ii) equilibrium efficiency. Our focus in this work is entirely on (i) and we consider pure Nash equilibria; however it should be noted that other equilibrium concepts are also of interest [25], [1], [13]. Recall that a *(pure Nash) equilibrium* is an action profile $a^* \in \mathcal{A}$ such that for each player i, $U_i(a_i^*, a_{-i}^*) = \max_{a_i \in \mathcal{A}_i} U_i(a_i, a_{-i}^*)$.

The Shapley value [24], one of the oldest and most commonly studied distribution rule in the cost sharing literature, is defined as

$$f^{SV}(i,S) = \sum_{T \subseteq S \setminus \{i\}} \frac{(|T|)!(|S|-|T|-1)!}{|S|!}(W(S \cup \{i\}) - W(S)).$$

The importance of the Shapley value is that it is budget-balanced and guarantees equilibrium existence in any game, regardless of its parameters. Further, it has many other desirable properties, e.g., it results in the game being a so-called "potential game" [26]. However, it has one key drawback – computing it is often intractable since it requires the calculation of exponentially many marginal contributions [7].

There are generalizations of the Shapley value that maintain its properties. In particular, the weighted Shapley value [23], which is defined as ($\boldsymbol{\omega}$ is the vector of the player-specific weights)

$$f^{WSV}(i,S;\boldsymbol{\omega}) = \sum_{T \subseteq S : i \in T} \frac{\omega_i}{\sum_{j \in T} \omega_j} \left(\sum_{R \subseteq T} (-1)^{|T|-|R|} W(R) \right),$$

also guarantees equilibrium existence in any game.

III. Results and Discussion

In the previous section, we discussed examples of budget-balanced distribution rules that guarantee equilibrium existence in "welfare" (cost or revenue) sharing games. The goal of this paper is to characterize the space of all such distribution rules.

Towards this end, this paper builds on the recent work of Chen, Roughgarden, & Valiant [6] and Marden & Wierman [17], which takes the first steps toward providing such a characterization. The following result combines the main contributions of [6] and [17] into one statement. Let $\mathcal{G}(N,f)$ denote the class of all games with a fixed player set N and budget-balanced distribution rule f. Note that this is a very general class; in particular, it includes games with arbitrary action sets and an arbitrary welfare function.

Theorem 1 *[6], [17] All games in $\mathcal{G}(N,f)$ possess a pure Nash equilibrium if and only if f is a weighted Shapley value.*

Less formally, Theorem 1 states that if one wants to use a distribution rule that guarantees equilibrium existence for all possible welfare functions and for all possible action sets, then one is limited to the class of weighted Shapley value distribution rules. Though this result seems quite general, it is shown by exhibiting a specific "worst-case" welfare function for which this limitation holds. In reality, when designing a distribution rule, one *knows* the specific welfare function for the situation, and Theorem 1 claims nothing in this case. In particular, there may be other distribution rules that guarantee equilibrium existence for all games, and recent work has shown that there are settings where this is the case [16].

Our main result shows that even for a fixed strictly submodular welfare function, the conclusion of Theorem 1 is still valid, i.e., weighted Shapley distribution rules are the only ones which guarantee equilibrium existence.

More specifically, let $\mathcal{G}(N,f,W)$ denote the class of all games with a fixed player set N, budget-balanced base distribution rule f, and welfare function W. Note that $\mathcal{G}(N,f,W) \subsetneq \mathcal{G}(N,f)$ because we have fixed the welfare function W (though arbitrary action sets are still allowed). We focus on welfare functions W that are *strictly submodular*, i.e., for player sets $X, Y \subseteq N$, $W(X) + W(Y) > W(X \cup Y) + W(X \cap Y)$. A variety of problems such as power control and coverage problems in sensor networks [4], [16], wireless access point assignment and frequency selection [11], and influence maximization [12] all have submodular welfare functions.

We can now state our main result.

Theorem 2 *Let W be a strictly submodular welfare function. All games in $\mathcal{G}(N,f,W)$ possess a pure Nash equilibrium if and only if f is a weighted Shapley value.*

Though Theorem 2 and Theorem 1 are superficially very similar, Theorem 2 is much stronger. The key contrast between Theorems 1 and 2 is that Theorem 1 states that there exists a welfare function for which the distribution rule is required to be a weighted Shapley value in order to guarantee equilibrium existence, while Theorem 2 states that, *for any strictly submodular welfare function*, the distribution rule must be a weighted Shapley value to guarantee equilibrium existence. To highlight this, consider that the proof of Theorem 1 exhibits a welfare function, and shows the result for that specific case, while the proof of Theorem 2 allows working with an arbitrary strictly submodular welfare function.

One subtle implication of Theorem 2 is that if one hopes to use a distribution rule that always guarantees equilibrium existence in games with a strictly submodular welfare function, then one is limited to working within the class of "potential games", since weighted Shapley value distribution rules result in potential games [26]. This is, perhaps, surprising since a priori potential games are often thought to be a small, special case of games. However, this is useful since there are many well understood learning dynamics which guarantee convergence to equilibria in potential games [3], [14], [15].

Theorem 2 also has some negative implications. First, the limitation to weighted Shapley value distribution rules means that one is forced to use distribution rules which are often intractable [7], as discussed earlier. Second, Marden & Wierman [17] show that there are efficiency limits that hold for any weighted Shapley value distribution rule. In particular, under any weighted Shapley value distribution rule there exists a game where the best equilibrium has welfare that is a multiplicative factor of two worse than the optimal welfare.

We do not have space to provide the complete proof of Theorem 2, so we sketch an outline, highlighting the proof technique and the key steps involved, in the following.

Proof Sketch of Theorem 2:

First, note that we only need to prove one direction since it is well known that a weighted Shapley value distribution rule is budget-balanced and guarantees equilibrium existence in any resource allocation game [22], [10], [18]. Thus, in the remainder of this section, we discuss the proof technique for the other direction – for budget-balanced distribution rules that are not weighted Shapley values, there exists a game for which no equilibrium exists.

The general outline of the proof is as follows. We establish several necessary conditions for a budget-balanced distribution rule f that guarantees the existence of an equilibrium for all games in $\mathcal{G}(N, f, W)$. Effectively, these necessary conditions eliminate any budget-balanced distribution rule that is not a weighted Shapley value and hence give us our desired result. We establish these conditions by a series of counterexamples which amount to choosing a resource set R, the respective values $\{v_r\}_{r\in R}$, and the associated action sets $\{\mathcal{A}_i\}_{i\in N}$.

A key technique of the proof is that instead of working with W directly, we define a basis of simple welfare functions, and represent W using this basis, i.e., any W is equivalent to a linear combination of the basis welfare functions. The basis we use is the following class of T-welfare functions. For every player subset $T \subseteq N$, a T-welfare function is defined as:

$$W^T(S) = \begin{cases} 1, & T \subseteq S; \\ 0, & \text{otherwise.} \end{cases} \tag{1}$$

It can be shown that the set of all T-welfare functions forms a basis for the set of all welfare functions, i.e., given any welfare function W, there exists a set $\mathcal{T} \subseteq 2^N$, and a sequence $Q = \{q_T\}_{T\in\mathcal{T}}$ of non-zero weights indexed by $\mathcal{T}$, such that:

$$W = \sum_{T\in\mathcal{T}} q_T W^T$$

From here on, we denote a welfare function by $(\mathcal{T}, Q)$, so the class of games defined in the theorem is denoted as $\mathcal{G}(N, f, \mathcal{T}, Q)$. It is useful to think of the sets in $\mathcal{T}$ as being "coalitions" of players that contribute to the welfare function, and the corresponding coefficients in Q as being their respective contributions. Also, for simplicity, we assume that $W(\emptyset) = 0$ and therefore, $\emptyset \notin \mathcal{T}$.

Warmup, a single T-welfare function: To build some intuition, we first present the proof outline for an isolated T-welfare function, i.e., for a W where $|\mathcal{T}| = 1$ above. This step consists of establishing the following necessary conditions for a budget-balanced f for any subset $S \subseteq N$ of players:

(a) If the coalition T is not formed in S ($T \nsubseteq S$), then f does not allocate any utility to the players in S.
(b) If the coalition T is formed in S ($T \subseteq S$), then f distributes the resulting welfare only among the contributing players (players in T). Therefore, players in $S - T$, if any, get nothing.
(c) If the coalition T is formed in S ($T \subseteq S$), then f distributes the welfare among players in T as if all other players (players in $S - T$) were absent.

It is easy to see that any budget-balanced f that satisfies the above conditions is completely specified by $|T| - 1$ values, namely the values of $f(i, T)$ for any $|T| - 1$ players in T. Further, it can be shown that $f(i, T)f(j, T) \geq 0$ for all $i, j \in T$. The proof is complete by observing that such an f is indeed a weighted Shapley value distribution rule, where the weight of player $i \in T$ is given by $\omega_i = \frac{f(i,T)}{q_T}$, and the weights of the other players are arbitrary.

To provide an idea of the proof technique we use to establish the above necessary conditions, in the following we prove (a). *Proof of (a):* Formally, we need to prove that any budget-balanced f that guarantees equilibrium existence in any game $G \in \mathcal{G}(N, f, \{T\}, \{q_T\})$ satisfies $f(i, S) = 0$ for all $i \in S$ when $T \nsubseteq S \subseteq N$. We do so by induction on $|S|$. The base case, where $|S| = 1$ is trivially true, because from budget balance, we get that for any player $i \in N$,

$$f(i, \{i\}) = \begin{cases} q_T & , T = \{i\} \\ 0 & , \text{otherwise} \end{cases}$$

For the induction hypothesis, let us assume that $(\forall\, S)$ $(\forall\, i \in S)$ $f(i, S) = 0$, where $T \nsubseteq S \subseteq N$ and $|S| = z$, for some integer z satisfying $0 < z < |N|$. Now, for $|S| = z + 1$, assume the contrary, that $f(i, S) \neq 0$ for some $i \in S$, where $T \nsubseteq S \subseteq N$ and $|S| = z+1$. Since f is budget-balanced, there has to be at least one other player $j \in S$ that also satisfies this condition, such that $f(i, S)f(j, S) < 0$, that is, $f(i, S)$ and $f(j, S)$ have opposite signs. Without loss of generality, assume that $f(i, S) < 0$ and $f(j, S) > 0$. From the induction hypothesis, we know that $f(i, S - \{j\}) = f(j, S - \{i\}) = 0$.

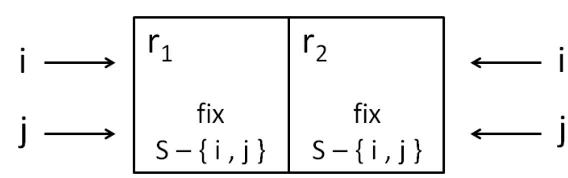

Fig. 1. Counter-example for (a)

Now consider the game illustrated in Fig. 1 with resource set $R = \{r_1, r_2\}$ and local resource coefficients $v_{r_1} = v_{r_2} > 0$. Players i and j have the same action sets – they can each choose either r_1 or r_2. All other players in S have a fixed action – they choose both resources. This is essentially a game between players i and j. It is easy to see that none of the four possible action profiles is a Nash equilibrium, which is a contradiction. For example, (r_1, r_1) is not a Nash equilibrium since $f(i, S - \{j\}) > f(i, S)$, and so player i has an incentive to deviate to r_2. This completes the inductive argument. $\square$

General welfare functions: We are now ready to outline the sequence of necessary conditions that make up the core of the proof for a general welfare function, $(\mathcal{T}, Q)$. Before continuing, we need to introduce some more notation. For any subset $S \subseteq N$, we denote by $\mathcal{T}(S)$ the set of all sets in $\mathcal{T}$ that are contained in S. That is, $\mathcal{T}(S) = \{T \in \mathcal{T} \mid T \subseteq S\}$. In other words, $\mathcal{T}(S)$ is the set of contributing coalitions in S. Also, let $\mathcal{I}(S)$ denote the set of players within $\mathcal{T}(S)$, that is, $\mathcal{I}(S) = \bigcup \mathcal{T}(S)$. In other words, $\mathcal{I}(S)$ is the set of contributing players in S.

The first three conditions mirror those stated above for an isolated T-welfare function – for any subset $S \subseteq N$ of players,

(a) If no coalition is formed in S ($\mathcal{T}(S) = \emptyset$), then f does not allocate any utility to the players in S.
(b) If any coalition is formed in S ($\mathcal{T}(S) \neq \emptyset$), then f distributes the welfare only among players in $\mathcal{I}(S)$.
(c) If any coalition is formed in S ($\mathcal{T}(S) \neq \emptyset$), f distributes the resulting welfare among players in $\mathcal{I}(S)$ as if all other players (players in $S - \mathcal{I}(S)$) were absent.

It turns out that in the general case, these three conditions are not enough to complete the proof.

We next show that the conditions above imply that f can now be represented as a linear combination of basis weighted Shapley value distribution rules – for every player subset $T \subseteq N$, define a basis T-distribution rule (parameterized by a positive vector $\boldsymbol{\omega}^T = \left(\omega_i^T\right)_{i \in T}$) as follows:

$$f^T(i, S; \boldsymbol{\omega}^T) = \begin{cases} \frac{\omega_i^T}{\sum_{k \in T} \omega_k^T} & , \ i \in T \text{ and } T \subseteq S \\ 0 & , \ \text{otherwise} \end{cases} \quad (2)$$

It is easy to see that (2) is the weighted Shapley distribution rule for its corresponding T-welfare function defined in (1), where the player weights are given by $\boldsymbol{\omega}^T$. Formally, we show that for any budget-balanced distribution rule f that guarantees equilibrium existence in any game $G \in \mathcal{G}(N, f, \mathcal{T}, Q)$, there exists a sequence of weight-vectors $\Omega = \left\{\boldsymbol{\omega}^T\right\}_{T \in \mathcal{T}}$ such that

$$f = \sum_{T \in \mathcal{T}} q_T f^T \quad (3)$$

where each f^T is a weighted Shapley value distribution rule for its corresponding T-welfare function, with weight vector $\boldsymbol{\omega}^T$. Therefore, f is completely specified by a sequence of weight vectors Ω.

Note that this is not yet enough to guarantee equivalence to a weighted Shapley value distribution rule, since a pair of players can have "inconsistent" weights in different coalitions. Thus, our final steps focus on deriving necessary consistency conditions for the weight vector Ω.

To guarantee consistency of the weights, we first show that if there is a pair of players common to two coalitions, then their weights in those two coalitions must be "consistent", i.e., they should be linearly dependent. More formally, we show that for every pair of subsets $T_1, T_2 \in \mathcal{T}$, every pair of players (i, j) such that $\{i, j\} \subseteq T_1 \cap T_2$,

$$\frac{\omega_i^{T_1}}{\omega_i^{T_2}} = \frac{\omega_j^{T_1}}{\omega_j^{T_2}}$$

Finally, we show that all weight vectors in Ω must be consistent, i.e., there exists a global weight vector $\boldsymbol{\omega} = (\omega_k)_{k \in N}$, such that, for every $T \in \mathcal{T}$, $\boldsymbol{\omega}^T$ and $\boldsymbol{\omega}$ restricted to T are linearly dependent. Note that we use the fact that the welfare function is strictly submodular only for proving this final step.

From the form (3) for the distribution rule f, and the form (2) for the basis T-distribution rule, it is clear that scaling the local weight vectors by a constant does not affect the distribution rule. Therefore, it follows from our final result that any budget-balanced distribution rule f that guarantees the existence of an equilibrium in any game $G \in \mathcal{G}(N, f, W)$, where W is strictly submodular, is completely specified by a global weight vector $\boldsymbol{\omega}$. The proof is complete by observing that, at this stage, f, in the form (3), is exactly the weighted Shapley value distribution rule for the welfare function $(\mathcal{T}, Q)$, with weight vector $\boldsymbol{\omega}$. □

References

[1] S. Adlakha, R. Johari, G. Weintraub, and A. Goldsmith. Oblivious equilibrium: an approximation to large population dynamic games with concave utility. In *Game Theory for Networks*, pages 68–69, 2009.
[2] E. Anshelevich, A. Dasgupta, J. Kleinberg, É. Tardos, T. Wexler, and T. Roughgarden. The price of stability for network design with fair cost allocation. In *FOCS*, pages 295–304, 2004.
[3] L. E. Blume. The statistical mechanics of strategic interaction. *Games and Economic Behavior*, 5(3):387–424, 1993.
[4] E. Campos-Náñez, E. Garcia, and C. Li. A game-theoretic approach to efficient power management in sensor networks. *Operations Research*, 56(3):552–561, 2008.
[5] M. Charikar, H. Karloff, C. Mathieu, J. S. Naor, and M. Saks. Online multicast with egalitarian cost sharing. In *SPAA*, pages 70–76, 2008.
[6] H.-L. Chen, T. Roughgarden, and G. Valiant. Designing network protocols for good equilibria. *SIAM J. Comput.*, 39(5):1799–1832, 2010.
[7] V. Conitzer and T. Sandholm. Computing shapley values, manipulating value division schemes, and checking core membership in multi-issue domains. In *AAAI*, 2004.
[8] J. Feigenbaum, C. H. Papadimitriou, and S. Shenker. Sharing the cost of multicast transmissions. *Comput. and Syst. Sciences*, 63(1):21–41, 2001.
[9] R. Gopalakrishnan, J. R. Marden, and A. Wierman. An architectural view of game theoretic control. *SIGMETRICS Performance Evaluation Review*, 38(3):31–36, 2011.
[10] S. Hart and A. Mas-Colell. Potential, value, and consistency. *Econometrica*, 57(3):589–614, 1989.
[11] B. Kauffmann, F. Baccelli, A. Chaintreau, V. Mhatre, K. Papagiannaki, and C. Diot. Measurement-based self organization of interfering 802.11 wireless access networks. In *IEEE INFOCOM*, pages 1451–1459, 2007.
[12] D. Kempe, J. Kleinberg, and E. Tardos. Maximizing the spread of influence through a social network. In *ACM KDD*, pages 137–146, 2003.
[13] J. R. Marden. State based potential games. *Under submission*.
[14] J. R. Marden, G. Arslan, and J. S. Shamma. Joint strategy fictitious play with inertia for potential games. *Transactions on Automatic Control*, 54(2):208–220, Feb 2009.
[15] J. R. Marden and J. S. Shamma. Revisiting log-linear learning: Asynchrony, completeness and a payoff-based implementation. *Under submission*.
[16] J. R. Marden and A. Wierman. Distributed welfare games. *Under submission*.
[17] J. R. Marden and A. Wierman. Limitations of utility design for multiagent systems. *Under submission*.
[18] D. Monderer and L. S. Shapley. Potential games. *Games and Economic Behavior*, 14(1):124–143, 1996.
[19] H. Moulin. Axiomatic cost and surplus sharing. In K. J. Arrow, A. K. Sen, and K. Suzumura, editors, *Handbook of Social Choice and Welfare*, volume 1, pages 289–357. Elsevier, 2002.
[20] H. Moulin and S. Shenker. Serial cost sharing. *Econometrica*, 60(5):1009–1037, 1992.
[21] H. Moulin and S. Shenker. Strategyproof sharing of submodular costs: Budget balance versus efficiency. *Econ. Theory*, 18(3):511–533, 2001.
[22] R. W. Rosenthal. A class of games possessing pure-strategy nash equilibria. *International Journal of Game Theory*, 2(1):65–67, 1973.
[23] L. S. Shapley. *Additive and non-additive set functions*. PhD thesis, Department of Mathematics, Princeton University, 1953.
[24] L. S. Shapley. A value for n-person games. In *Contributions to the theory of games – II*. Princeton University Press, 1953.
[25] Y. Su and M. van der Schaar. Conjectural equilibrium in multiuser power control games. *Trans. on Signal Processing*, 57(9):3638–3650, 2009.
[26] T. Ui. Shapley value representation of potential games. *Games and Economic Behavior*, 31(1):121–135, 2000.
[27] D. H. Wolpert and K. Tumer. An introduction to collective intelligence. In *Handbook of Agent technology. AAAI*, 1999.
[28] H. P. Young. *Equity: In theory and practice*. Princeton University Press, 1994.